Methods for analyzing realistic, compact sources

Techniques we have studied in depth have relied upon either approximations to the full field equations, or have assumed special symmetry.

→ Surprisingly robust! Weak field limit encompasses a very wide range of situations that are encountered in real astrophysical environments - gravitational lensing, solar system, many binary stars; large scale structure of universe, black hole solutions.

Though robust, we can do better! Two approaches in this lecture: ① Iterating from weak to not-so-weak fields

② Examining perturbations about our exact solutions (focus on black holes; moral equivalent in cosmological contexts).

Next lecture: direct numerical solution of EFEs.

For ②: see L. Rezzolla, gr-qc/0302025

Iteration from weak field: Post-Newtonian theory.

Reference: Luc Blanchet, Living Reviews of Relativity, v 17, p2 (2014)

Einstein field equation normally written

$$G^{\alpha\beta}[g, \partial g, \partial^2 g] = 8\pi G\, T^{\alpha\beta}[g]$$

Bianchi identity $\nabla_\alpha G^{\alpha\beta} = 0$ implies source conservation, $\nabla_\alpha T^{\alpha\beta} = 0$.

Post-Newtonian theory begins by defining a variable that looks like a metric perturbation:

$$h^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$$

ie, $g^{\alpha\beta} = (-g)^{-1/2}\left(\eta^{\alpha\beta} + h^{\alpha\beta}\right)$

<u>NOT</u> assuming $||h^{\alpha\beta}|| \ll 1$!

We impose one condition on $h^{\alpha\beta}$: $\partial_\alpha h^{\alpha\beta} = 0$
"deDonder gauge" or "harmonic coordinates"

With this in place, the <u>exact</u> Einstein field equations become

$$\Box h^{\alpha\beta} = 16\pi G \tau^{\alpha\beta}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ — <u>flat</u> spacetime wave operator!

$$\tau^{\alpha\beta} = (-g)T^{\alpha\beta} + \frac{\Lambda^{\alpha\beta}}{16\pi G}$$

$\Lambda^{\alpha\beta}$ = nonlinearities in h (see slide)

$$= N^{\alpha\beta}[h,h] + M^{\alpha\beta}[h,h,h] + L^{\alpha\beta}[h,h,h,h] + \ldots$$

Formal solution is trivial using Green's function for $\Box$:

$$h^{\alpha\beta}(\underset{\sim}{x},t) \equiv -4G\int \frac{d^3x'}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\,\tau^{\alpha\beta}(\underset{\sim}{x}';t-|\underset{\sim}{x}-\underset{\sim}{x}'|)$$

Exact for any source ... but apparently useless, as it's an integro-differential equation with the solution we seek, $h^{\alpha\beta}$, buried on the RHS in $\tau^{\alpha\beta}$.

Path to enlightenment: RHS introduces "small parameter" G.

Perturbation theory: conceptually similar to weak field analysis.

$$g_{\alpha\beta} = g^B_{\alpha\beta} + h_{\alpha\beta}, \quad \|h_{\alpha\beta}\| / \|g^B{}_{\alpha\beta}\| \ll 1.$$

Expand Einstein to linear order in h, develop a "wave equation" for the perturbation.

Consider vacuum perturbations of Schwarzschild:

$$^{(B)}ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Vacuum means $G_{\alpha\beta} = 0 \rightarrow R_{\alpha\beta} = 0$.

$$\rightarrow \hat{R}_{\alpha\beta} + \delta R_{\alpha\beta} = 0$$

$\hat{R}_{\alpha\beta}$ $\rightarrow$ automatically satisfied since background is a vacuum solution.

$$\rightarrow \delta R_{\alpha\beta} = 0 \quad \text{is our wave equation.}$$

$$\rightarrow \hat{\nabla}_\beta \left(\delta\Gamma^\gamma{}_{\alpha\gamma}\right) - \hat{\nabla}_\gamma \left(\delta\Gamma^\gamma{}_{\alpha\beta}\right) = 0$$

$$\delta\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}_B \left(\hat{\nabla}_\beta h_{\alpha\delta} + \hat{\nabla}_\alpha h_{\beta\delta} - \hat{\nabla}_\delta h_{\alpha\beta}\right)$$

Before grinding, pause & examine symmetries. Since Schw. is spherically symmetric, should find a harmonic expansion to be useful:

$$h_{\alpha\beta} = \begin{cases} h_{00} & \text{- scalar harmonics} \\ h_{0i} & \text{- vector harmonics} \\ h_{ij} & \text{- tensor harmonics.} \end{cases}$$

Vector & tensor harmonics will come in two different parity varieties:

Example: $$h_{ij} = \sum_{lm} \left[a_{lm} \underset{\uparrow\,\text{even}}{A^{lm}_{ij}} (\theta, \phi) + b_{lm} \underset{\uparrow\,\text{odd}}{B^{lm}_{ij}} (\theta, \phi) \right]$$

$$P(\mathcal{J}(\theta, \phi)) = \mathcal{J}(\pi - \theta, \phi + \pi)$$
$$= (-1)^l \mathcal{J}(\theta, \phi) \quad \text{"even", "polar"}$$
$$= (-1)^{l+1} \mathcal{J}(\theta, \phi) \quad \text{"odd", "axial".}$$

Focus on odd parity. Each mode (l, m) contributes as

$h_{00} = 0$ (Spherical harmonics are polar!) (even)

$$h_{0i} = H_0(t, r) \left[0, -\csc\theta\, \partial_\phi Y_{lm}, \sin\theta\, \partial_\theta Y_{lm} \right]$$

$$h_{ij} = H_1(t, r) [\hat{e}_1]_{ij} + H_2(t, r) [\hat{e}_2]_{ij}$$

$$[\hat{e}_1]_{ij} = \begin{bmatrix} 0 & -\csc\theta\, \partial_\phi Y_{lm} & \sin\theta\, \partial_\theta Y_{lm} \\ -\csc\theta\, \partial_\phi Y_{lm} & 0 & 0 \\ \sin\theta\, \partial_\theta Y_{lm} & 0 & 0 \end{bmatrix}$$

$[\hat{e}_2]_{ij}$ = mess involving 2nd derivs of Y_{lm}.

2 Further simplifications:

1. Focus on $m=0 \rightarrow \partial_\phi = 0$. Not necessary - axisymmetry means get same wave equation for all m. But cleans up.

2. Set $H_2(t, r) = 0 \rightarrow$ "Regge-Wheeler gauge".

Result: $$h_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & H_0 \\ 0 & 0 & 0 & H_1 \\ 0 & 0 & 0 & 0 \\ H_0 & H_1 & 0 & 0 \end{bmatrix} \sin\theta \, \partial_\theta P_\ell$$

Legendre polynomial

Run this through $\delta R_{\alpha\beta} = 0$. Result:

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} + \left(1 - \frac{2GM}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6GM}{r^3}\right] Q = 0$$

$$Q = \frac{H_1}{r}\left(1 - \frac{2GM}{r}\right)$$

$$\frac{\partial H_0}{\partial t} = \frac{\partial}{\partial r_*}\left(r_* Q\right)$$

$$r_* = r + 2GM \ln\left[\frac{r}{2GM} - 1\right]$$

"Regge-Wheeler" equation. Same exercise for even parity modes - Zerilli equation. (Similar, but messier potential.)

Note asymptotic solutions: $\dfrac{\partial^2 Q}{\partial t^2} - \dfrac{\partial^2 Q}{\partial r_*^2} = 0$

for $r \to \infty$, $r \to 2GM$: $Q \sim \exp\left[i\omega\left(t \pm r_*\right)\right]$

Add physical boundary conditions:

$Q \sim \exp\left[i\omega\left(t - r_*\right)\right]$ $r \to \infty$ $\to$ OUTGOING

$Q \sim \exp\left[i\omega\left(t + r_*\right)\right]$ $r \to 2GM$ $\to$ INGOING.

Can find modes which asymptote to this limits, but are good at all r - "Quasi-normal modes".
Represent oscillation of black hole geometry.

Solution takes form $\omega = \omega_r + i\omega_i$

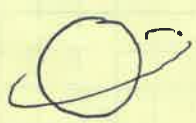
For $l=2$, $\omega_r \simeq \frac{0.37}{M} = 7.5 \times 10^3\, s^{-1} \left(\frac{10 M_\odot}{M} \right)$

$\frac{1}{2}\frac{\omega_r}{\omega_i} \equiv$ Quality factor $\simeq 2$.

Fit Kerr results: $\omega_r \simeq \frac{1}{M}\left(1 - 0.63(1-a)^{3/10}\right)$

Quality $\simeq 2(1-a)^{-9/20}$!!

Can also use this if perturbation arises from matter source:



$$T_{\alpha\beta} = \mu u_\alpha u_\beta \, \delta^4\left[\vec{x} - \vec{z}(\tau)\right]$$

μ: rest mass of small body

Use homogeneous solution to make green's function, ~~a~~ integrate green over source.

Kerr? Unfortunately, metric equation doesn't work well - lack of symmetry & decent mode functions.

By a miracle, get "good" equations by focusing on <u>curvature</u>:

$$\cancel{\nabla^\epsilon [\nabla_\epsilon R_{\alpha\beta\gamma\delta} + \nabla_\delta R_{\alpha\beta\epsilon\gamma} + \nabla_\gamma R_{\alpha\beta\delta\epsilon}] = 0}$$

$$\nabla^\gamma [\nabla_\gamma R_{\alpha\beta\delta\epsilon} + \nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon}] = 0$$

Vacuum means $R_{\alpha\beta\delta\epsilon} \to C_{\alpha\beta\delta\epsilon}$

Choose a basis tetrad adapted to radiation:

$$l_\alpha \doteq \left[1, -\frac{\rho^2}{\Delta}, 0, -a\sin^2\theta\right] \quad \text{"ingoing"}$$

$$n_\alpha \doteq \frac{1}{2}\left[\frac{\Delta}{\rho^2}, 1, 0, \frac{-a\Delta\sin^2\theta}{\rho^2}\right] \quad \text{"outgoing"}$$

$$m_\alpha = \frac{\rho^{-1}}{\sqrt{2}}\left[ia\sin\theta, 0, \rho^2, -i(r^2+a^2)\sin\theta\right]$$

$\to$ "angular"

$$\rho = \left(\frac{-1}{r - ia\cos\theta}\right)^{-1}$$

Define 5 complex scalars:

$$\left.\begin{aligned} \Psi_0 &= -C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma m^\delta \\ \Psi_4 &= -C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta \end{aligned}\right\} \text{Radiative curvature degrees of freedom}$$

$\Psi_1, \Psi_2, \Psi_3 \to$ non-radiative.

Do perturbative expansion: $\Psi_x = \hat{\Psi}_x + \delta\Psi_x$

Plug in to Bianchi + grind.

Focus of Ψ_4: $\hat{\Psi}_4 = 0$. Find

$$\Psi_4 = \frac{1}{(r - ia\cos\theta)^4} \sum R_{\ell m}(r)\, S_{\ell m}(\theta)\, e^{im\phi}\, e^{-i\omega t}$$

$S_{\ell m}(\theta) \equiv$ "spin-weighted spheroidal harmonic"

$R_{\ell m}(r)$: solution of

$$\Delta^2 \frac{d}{dr}\left[\frac{1}{\Delta}\frac{dR}{dr}\right] - V_{\ell m} R = -\mathcal{T}(r)$$

"Teukolsky equation"

Source built from $T_{\alpha\beta}$ of matter or fields perturbing BH.