Perturbing Schwarzschild

$$ds^2 = \left(g^{\mathrm{B}}_{\alpha\beta} + h_{\alpha\beta}\right) dx^\alpha dx^\beta$$

Take background to be Schwarzschild, consider vacuum perturbations:

$$G_{\alpha\beta} = 0 \longrightarrow \delta R_{\alpha\beta} = 0$$

Build Ricci perturbation from $h_{\alpha\beta}$; break $h_{\alpha\beta}$ into scalar, vector, tensor pieces; characterize them by their parity properties.

Results for odd parity

$$h_{00} = 0$$

$$h_{0i} \doteq H_0(t,r)\left[0, -\csc\theta\partial_\phi Y_{\ell m}, \sin\theta\partial_\theta Y_{\ell m}\right]$$

$$h_{ij} = H_1(t,r)\mathbf{e}^1_{ij} + H_2(t,r)\mathbf{e}^2_{ij}$$

See Rezzolla, gr-qc/0302025 for details and the explicit form of the basis tensors (which include first and second derivatives of Y_{lm}).

Can put $m = 0$ (axisymmetry); can set $H_2 = 0$ by choice of gauge.

The Regge-Wheeler equation

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} + \left(1 - \frac{2GM}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6GM}{r^3}\right] Q = 0$$

$$r_* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

$$Q \equiv \frac{H_1}{r}\left(1 - \frac{2GM}{r}\right)$$

$$\frac{\partial H_0}{\partial t} = \frac{\partial}{\partial r_*}(r_* Q)$$

As $r \to \infty$ and $r \to 2GM$, this simplifies:

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} = 0$$ so $Q \sim e^{i\omega(t \pm r_*)}$ in this limit.

Minus sign corresponds to outgoing wave packet; plus sign to ingoing.

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Physics: Nothing can come out of the event horizon; nothing can come in from infinity.

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$$\frac{\partial H_0}{\partial t} = \frac{\partial}{\partial r_*}(r_* Q)$$

Require

$$Q \sim e^{i\omega(t - r_*)} \qquad r \to \infty$$

$$Q \sim e^{i\omega(t + r_*)} \qquad r \to 2GM$$

The Regge-Wheeler equation

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$$\frac{\partial H_0}{\partial t} = \frac{\partial}{\partial r_*}\left(r_* Q\right)$$

Require

$$Q \sim e^{i\omega(t - r_*)} \qquad r \to \infty$$

$$Q \sim e^{i\omega(t + r_*)} \qquad r \to 2GM$$

Turns into an eigenvalue problem: There exist special frequencies such that Q is purely ingoing on the horizon, purely outgoing at infinity.

Quasi-normal modes

Frequencies which do this depend on the angular index *l*, and have real and imaginary parts:

$$\omega = \omega_r + i\omega_i$$

Outgoing waves then have the form

$$Q \sim e^{-t/\tau} e^{i\omega_r(t - r_*)}\ , \qquad \tau = 1/\omega_i$$

Values of ω_r and τ in general need to be found numerically … l = 2 yields the longest lived modes:

$$\omega_r \simeq \frac{0.37}{GM} \quad \rightarrow \quad f_r \equiv \frac{\omega_r}{2\pi} = 240\,\mathrm{Hz}\left(\frac{50\,M_\odot}{M}\right)$$

$$\tau\omega_r \simeq 4$$

Kerr

Metric expansion doesn't work as well for Kerr since we no longer have spherical symmetry.

However, a minor miracle occurs if you perturb the *curvature*: Recall pset exercise in which you used Bianchi to make a wave equation for Riemann:

$$\nabla^\gamma \left[\nabla_\gamma R_{\alpha\beta\delta\epsilon} + \nabla_\alpha R_{\beta\gamma\delta\epsilon} + \nabla_\beta R_{\gamma\alpha\delta\epsilon}\right] = 0$$

The first term is a wave operator; applying identities becomes a wave equation with "Riemann squared" acting as a source term. Split Riemann:

$$R_{\alpha\beta\gamma\delta} = R^{\rm B}_{\alpha\beta\gamma\delta} + \delta R_{\alpha\beta\gamma\delta}$$

Vacuum; pick components

In vacuum, Ricci curvature vanishes; Riemann is equivalent to Weyl: $R_{\alpha\beta\gamma\delta} \rightarrow C_{\alpha\beta\gamma\delta}$.

Can also organize the components by projecting them onto a special set of basis vectors:

$$l^\alpha \doteq \frac{1}{\Delta}\left[(r^2+a^2), \Delta, 0, a\right]$$

(Tangent to outgoing null geodesics)

$$n^\alpha \doteq \frac{1}{2(r^2+a^2\cos^2\theta)}\left[(r^2+a^2), -\Delta, 0, a\right]$$

(Tangent to ingoing null)

$$m^\alpha \doteq \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left[ia\sin\theta, 0, 1, i\csc\theta\right]$$

(This plus complex conjugate cover the angular degrees of freedom)

Complex Weyl scalars

10 degrees of freedom in the Weyl curvature are given by the following 5 complex numbers:

$$\Psi_0 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta \quad \Psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$$

$$\Psi_1 = -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta \quad \Psi_3 = -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta$$

$$\Psi_2 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta$$

Introduce perturbative expansion for these quantities:

$$\Psi_n = \Psi_n^{\mathrm{B}} + \delta\Psi_n$$

Remake the Riemann wave equation in terms of $\delta\Psi_n$

Done by Teukolsky in 1973.

Result 1: Ringing modes with spin

Focus on Ψ_4. The resulting equation describes radiation far from the perturbed black hole. The equation that results turns out to separate:

$$\Psi_4 = \frac{1}{(r - ia\cos\theta)^4} \int d\omega \sum_{\ell m} R_{\ell m\omega}(r) S_{\ell m\omega}(\theta) e^{im\phi} e^{-i\omega t}$$

Fairly simple equations govern the behavior of *R* and *S*.

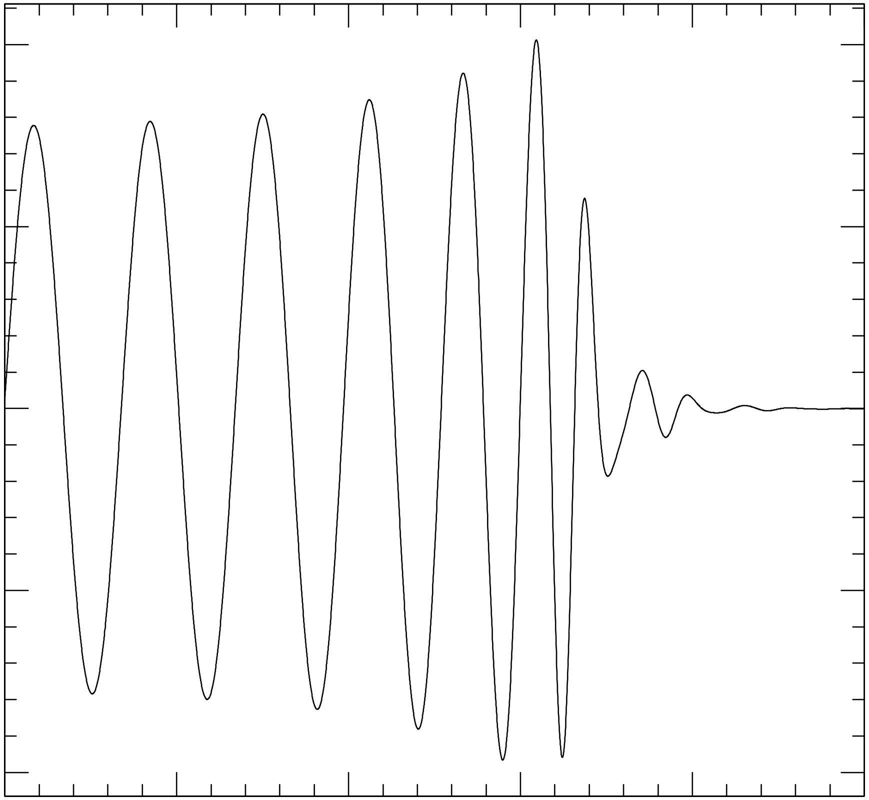
Again find special frequencies for which Ψ_4 is purely ingoing boundary at horizon, outgoing at infinity:

$$\Psi_4(r \to \infty) \sim e^{-t/\tau_{\ell m}} e^{i\omega_{\ell m} t}$$

$$\omega_{22} \approx \frac{1}{GM}\left[1 - 0.63(1-a)^{0.7}\right] \qquad \tau_{22}\omega_{22} \approx 4(1-a)^{-0.45}$$

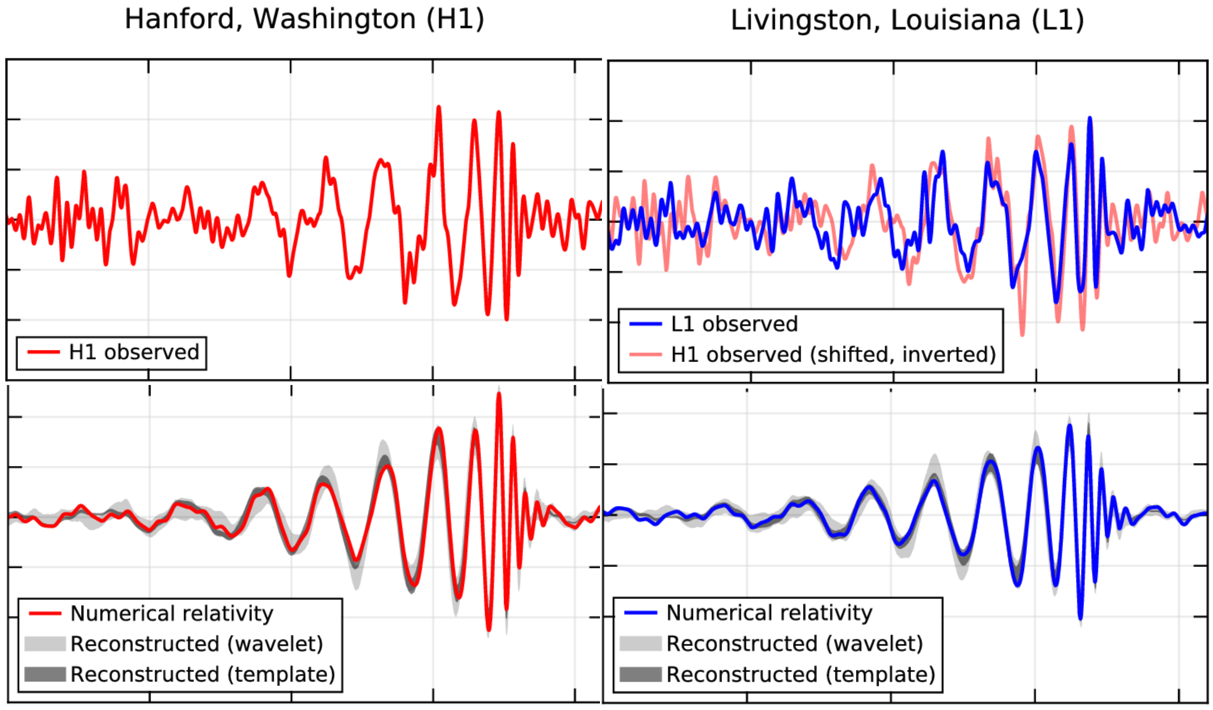
Example of ringdown

Example for a black hole with spin a = 0.8M



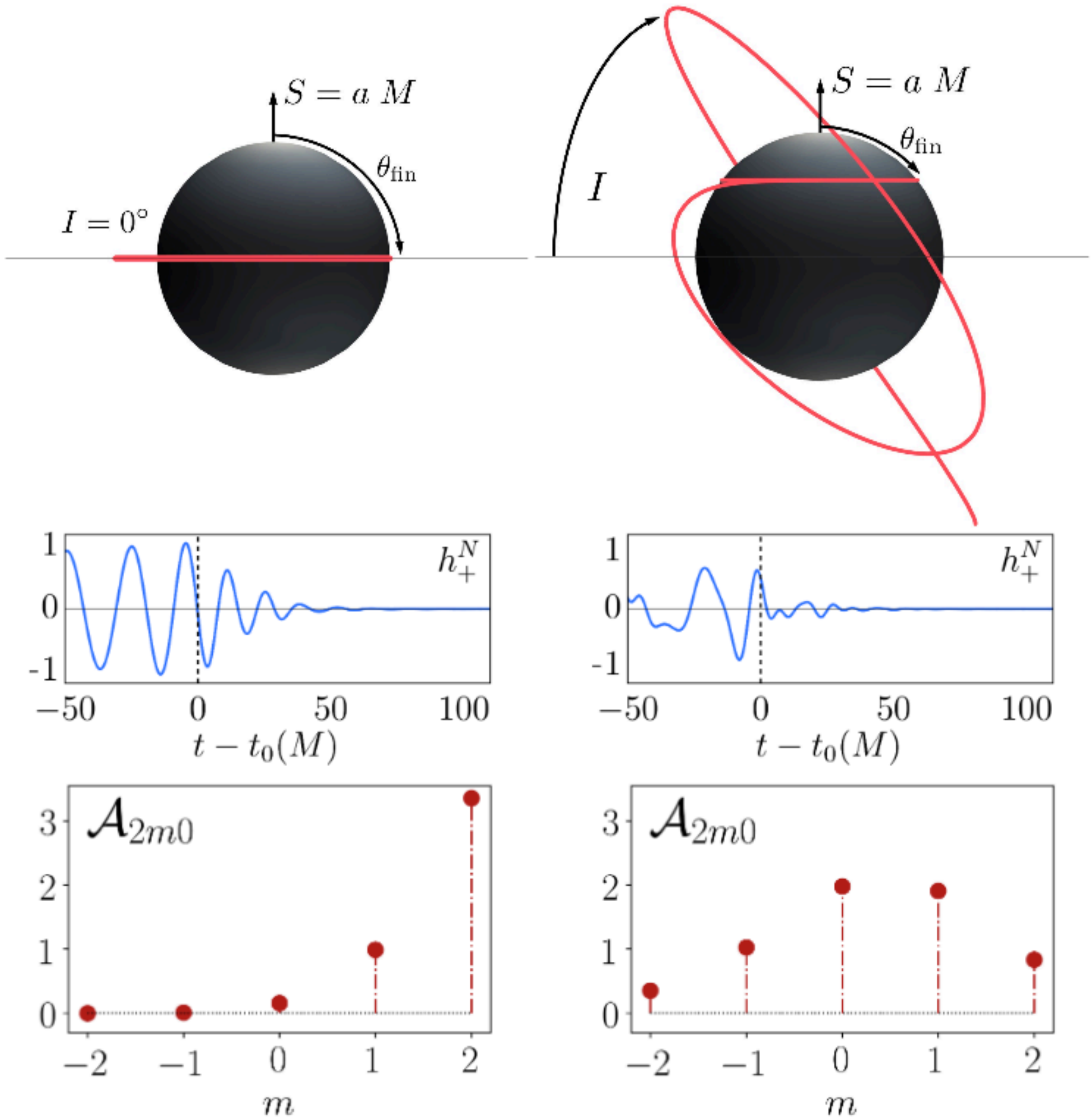
Frequency and damping time determined by — and thus encode — the mass and spin of the final black hole.

First event shows some of the best evidence of this structure



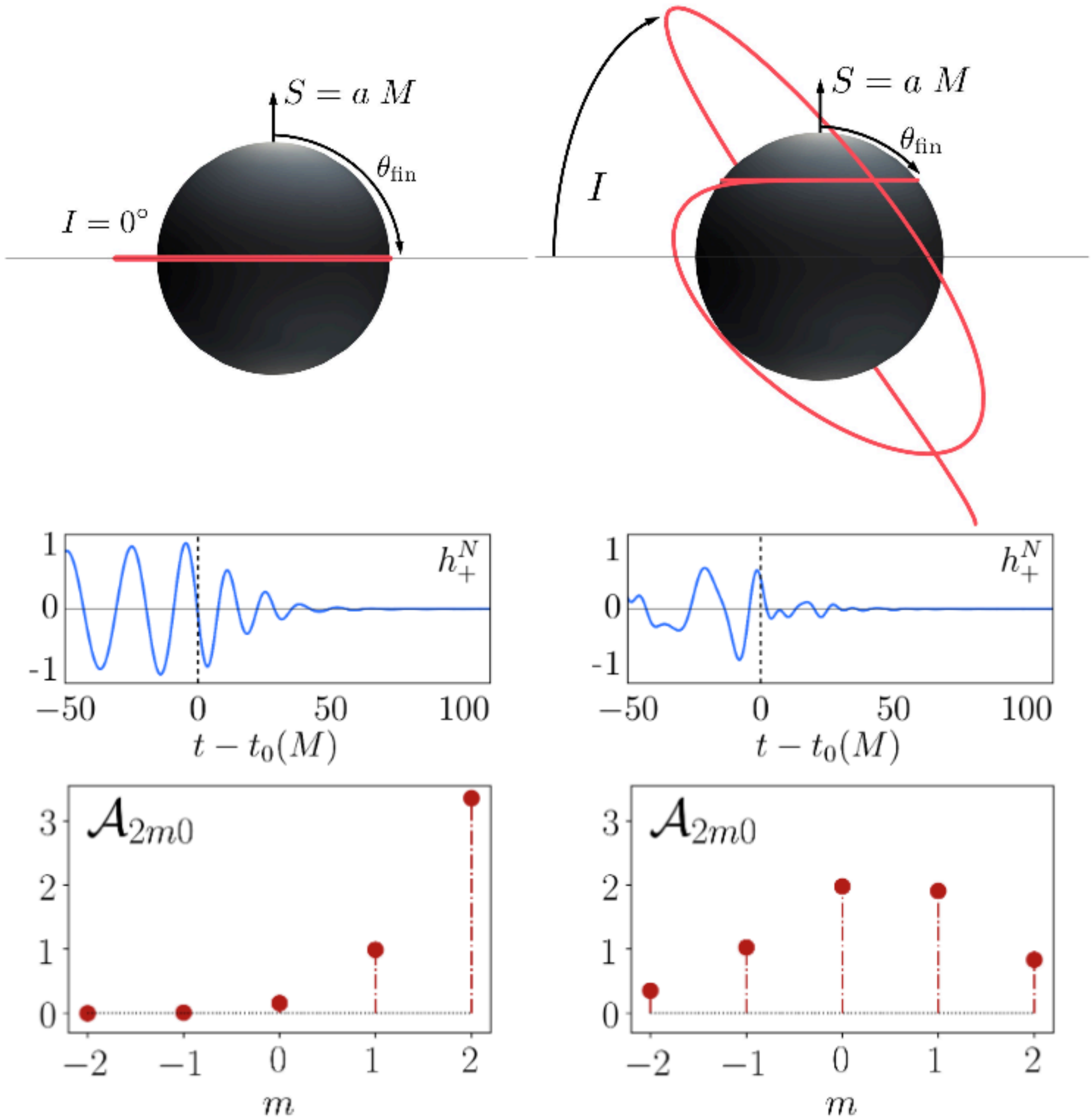
Last few cycles of GW150914 are consistent with this structure, for a final black hole with $a/M = 0.7$

More events will more fully explore parameter space



Example: Recent work from my group (arXiv:1901.05900; S.A. Hughes, A. Apte, G. Khanna, H. Lim), showing how the spectrum of mode excitation depending on the geometry of the final plunge.

More events will more fully explore parameter space



Advertisement: Halston Lim's PhD defense **tomorrow** at 9 am in the Marlar Lounge of the McNair building!

Can also include source … lets us study binaries with one member much larger than other.