Numerical relativity: directly solving Einstein's equations on a computer.

Recall an old homework exercise: Bianchi identity

$$\nabla^a G_{ab} = 0 \rightarrow \nabla^0 G_{0b} = -\nabla^i G_{ib}$$

NOTE: using "Fortran" convention in this lecture!

Can therefore only have 1 time derivative.

Has at most 2 time derivatives

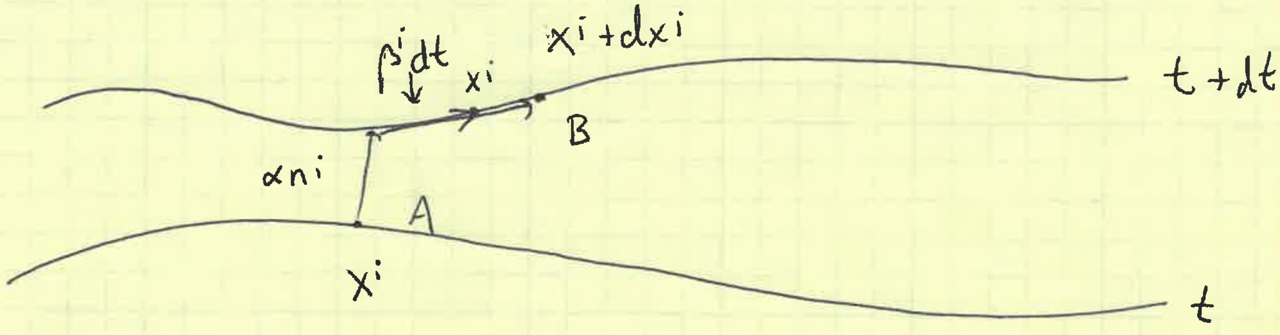
Allows a split into "constraint" and "evolution" equations:

$G_{0b} = 8\pi G T_{0b} \rightarrow$ Relates $g, \partial_t g$ at a single "moment of time"

$G_{ij} = 8\pi G T_{ij} \rightarrow$ relates $g, \partial_t g, \partial_t^2 g \rightarrow$ Turns into an evolution equation, stepping us from "one moment of time to another"

To do an evolution like this, need to <u>explicitly</u> choose a notion of time!

Choosing our coordinates forces us to introduce several auxiliary notions: a "lapse" and a "shift":



Event A is at (t, x^i)

Event B is at $(t + dt,\ x^i + dx^i)$

n^i is the <u>normal</u> to the "time slice".

The proper time experienced by an observer who moves along n^i from t to $t + dt$ is

$$d\tau = \alpha\, dt$$

↳ lapse converts coordinate interval to proper interval for "normal" observer.

→ α gives us the freedom to run time at different rates in different parts of our spacetime!

β^i is the shift: it reflects our freedom to slide spatial coordinates around however we want in ~~our~~ each of our time slices.

Notice that the total spacetime distance from pt A to B is

$$ds^2 = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

"Arnowitt, Deser, Misner" (ADM) form of the metric.

Exercise helped clarify what we need to do:

1. Pick spacetime coordinates $\longleftrightarrow$ pick α, β^i.
2. Figure out how to express all the geometric tensors of GR as quantities that "live" in the 3-dimensional slices.
3. Use this to break up $G_{ab} = 8\pi G T_{ab}$ into appropriate equations
4. Pick initial data and attack.

A bit more formally: Take our spacetime manifold M, "foliate" it with level surfaces of some scalar function t. Define a 1-form:

$$\Omega_a \equiv \nabla_a t$$

Define the normal of this 1-form via

$$g^{ab}\Omega_a\Omega_b \equiv -1/\alpha^2$$

(If Ω_a is timelike, then $\alpha > 0$.) Use this to normalize: $\omega_a \equiv \alpha\,\Omega_a$

[1 Useful property: $\omega_{[a}\nabla_b\,\omega_{c]} = 0$
"irrotational".]

Define the normal: $n^a \equiv -g^{ab}\,\omega_b$ so it points to increasing t.

so $n_a = -\omega_a$

$$n_a = g_{ab}n^b = -g_{ab}g^{bc}\omega_c = -\omega_c$$

Notice: $n^a n_a = +g^{ab}\,\omega_a\omega_b$

$$= \alpha^2 g^{ab}\Omega_a\Omega_b = -1$$

$\rightarrow$ useful to regard n^a as the 4-velocity of a particular preferred observer!

n^a defines the timelike direction. A useful quantity to make is the tensor which projects orthogonal to this direction:

$$\gamma_{ab} \equiv g_{ab} + n_a n_b$$

Can regard γ_{ab} as metric in slice!

Notice: $n^a \gamma_{ab} = n_b + (n_a n^a) n_b$
$= n_b - n_b = 0.$

Inverse: $\gamma^{ab} = g^{ac} g^{bd} \gamma_{cd}$
$= g^{ab} + n^a n^b$

Mixed: $\gamma^a{}_b = \delta^a{}_b + n^a n_b$

(Sometimes useful to have a tensor that projects into the timelike direction:

$$N^a{}_b \equiv - n^a n_b.)$$

Use projection tensor to define any tensor in slice:

$$[T^a{}_b]_{\text{in slice}} = \gamma^a{}_c \gamma^d{}_b T^c{}_d$$

In particular, use this to define covariant derivative in time slice:

$$D_a f \equiv \text{3-D covar deriv in slice}$$
$$= \gamma^b{}_a \nabla_b f$$
$$D_a [T^b{}_c]_{\text{in slice}} = \gamma^d{}_a \gamma^b{}_e \gamma^f{}_c \nabla_d T^e{}_f$$

Note: can show $D_a \gamma_{bc} = 0$

→ can define a 3-D christoffel in the usual way, use it to write "covar = partial + Γ."

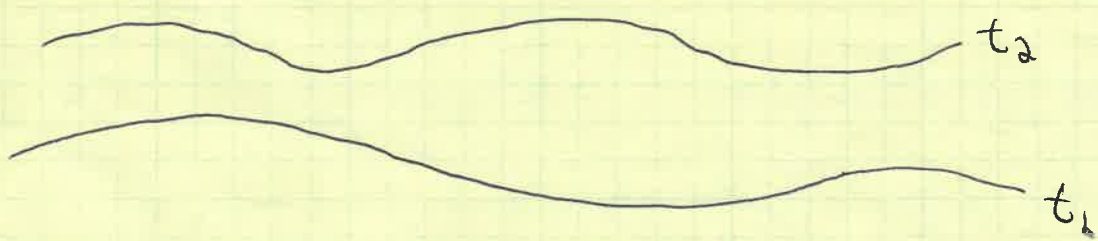
Last thing we need to do is define curvature tensors.

2 major pieces, plus a final step:

1. Riemann tensor intrinsic to the geometry of a single slice - usual formula, but assembled from the spatial metric γ_{ab}. Define in terms of commutator: $[D_a, D_b] w_c = R^d{}_{abc} w_d$

   $R^d{}_{abc} n_d = 0$. ↖ 3-D curvature!!

2. Curvature due to "embedding" of each timeslice in the 4-dimensional geometry:



This "extrinsic" curvature is clearly related to the expansion / divergence of normal vectors.

Define "Expansion": $\Theta_{ab} = \gamma^c{}_a \gamma^d{}_b \nabla_{(c} n_{d)}$

"twist": $\omega_{ab} = \gamma^c{}_a \gamma^d{}_b \nabla_{[c} n_{d]}$

$= 0$ due to irrotation.

Definition: "Extrinsic curvature" is the (negative) of the expansion:

$$K_{ab} = -\gamma^c{}_a \gamma^d{}_b \nabla_c n_d$$

no need to symmetrize, since antisymmetric piece vanishes.

Definition: acceleration: $a_c = n^b \nabla_b n_c$.

Using this, our definition becomes

$$K_{ab} = -\nabla_a n_b - n_a a_b$$

$$\equiv -\frac{1}{2} \mathcal{L}_{\vec{n}} \gamma_{ab}$$

"Extrinsic curvature is the time derivative of the spatial geometry".

Final groundwork: Relate the "real" 4-D curvature ${}^{(4)}R^a{}_{bcd}$ to the curvature on a slice & the extrinsic curvature.

Results:

1. Gauss's Equation

$$\gamma^p{}_a \gamma^q{}_b \gamma^r{}_c \gamma^s{}_d \, {}^{(4)}R_{pqrs} = R_{abcd} + 2K_{c[a}K_{b]d}$$

Fully spatial projection of ${}^{(4)}$ Riemann.

2. Codazzi Equation

$$\gamma^p{}_b \gamma^q{}_a \gamma^r{}_c \, n^s \, {}^{(4)}R_{rpqs} = 2D_{[a}K_{c]b}$$

One time component.

3. Ricci's Equation

$$\mathcal{L}_{\vec{n}} K_{ab} = n^d n^c \gamma^q{}_a \gamma^r{}_b \, {}^{(4)}R_{rdqc} - \frac{1}{\alpha} D_a D_b \alpha - K^c{}_b K_{ac}$$

$$n^d n^c \gamma^q{}_a \gamma^r{}_b \, {}^{(4)}R_{rdqc} = \frac{1}{\alpha} D_a D_b \alpha + K^c{}_b K_{ac} + \mathcal{L}_{\vec{n}} K_{ab}$$

Now, we have all the pieces - ready to attack!

$$^{(4)}G_{ab} = 8\pi G\, ^{(4)}T_{ab}$$

Look at 3 projections of this:

1 $$n^a n^b\, ^{(4)}G_{ab} = 8\pi G\, n^a n^b\, ^{(4)}T_{ab}$$
$$\equiv 8\pi G \rho$$

ρ = energy density measured by a "normal" observer

$$\rightarrow \boxed{R + K^2 - K_{ab}K^{ab} = 16\pi G\rho}$$

"Hamiltonian constraint"

$K = \gamma^{ab} K_{ab}$

2 $\gamma^a{}_c n^b$ (Einstein eq):

$$\rightarrow D_b K^b{}_a - D_a K = -8\pi G\, \gamma^b{}_a n^c\, ^{(4)}T_{bc}$$
$$\equiv 8\pi G\, j_a$$

$j_a \equiv$ momentum density measured by a normal observer.

3 Both spatial projections: 1st, define the time direction

$$t^a = \alpha n^a + \beta^a$$

$n^a \equiv$ "Eulerian observer", at rest in slice

$$\rightarrow t^a \Omega_a = 1.$$

$t^a \equiv$ "coordinate observer.

Dual to Ω_a.

$$\rightarrow \mathcal{L}_{\vec{t}} K_{ab} = -D_a D_b \alpha + \alpha\left(R_{ab} - 2K_{ac}K^c{}_b + K K_{ab}\right)$$
$$- 8\pi G \alpha\left(S_{ab} + -\tfrac{1}{2}\gamma_{ab}(S-\rho)\right)$$
$$+ \mathcal{L}_{\vec{\beta}} K_{ab}$$