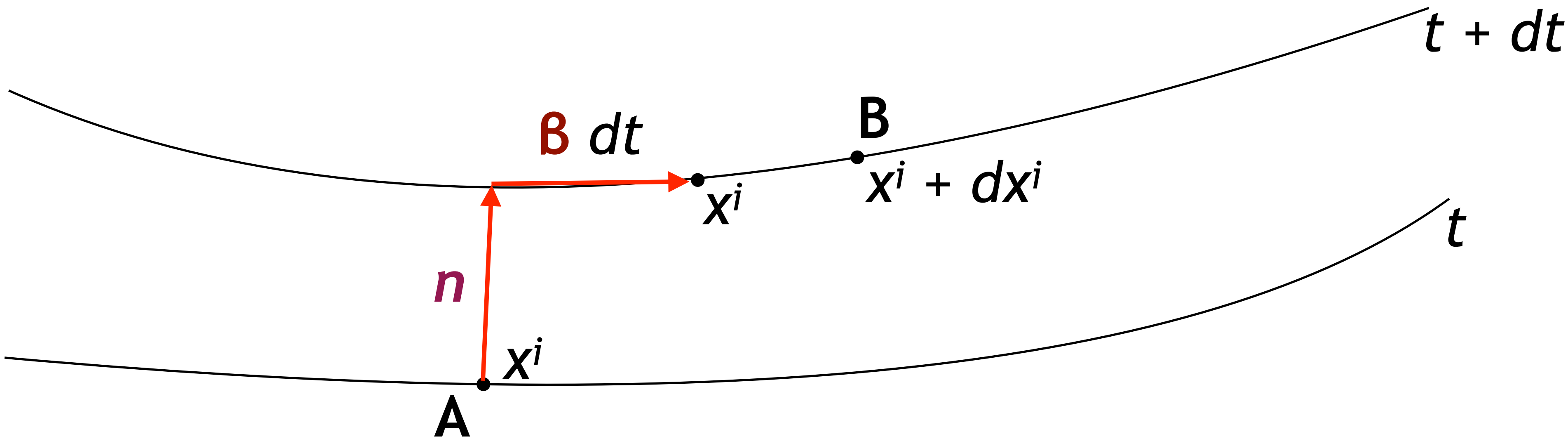
Split spacetime into space & time



Event **A**: (t, x^i) Event **B**: $(t + dt, x^i + dx^i)$

n is normal to "time slice." Proper time experienced by an observer who moves along n^i from t to $t + dt$ is $d\tau = \alpha\, dt$ … function α, the *lapse*, converts coordinate interval to proper interval for a "normal" observer.

Lapse lets us run time at different rates in different parts of our spacetime.

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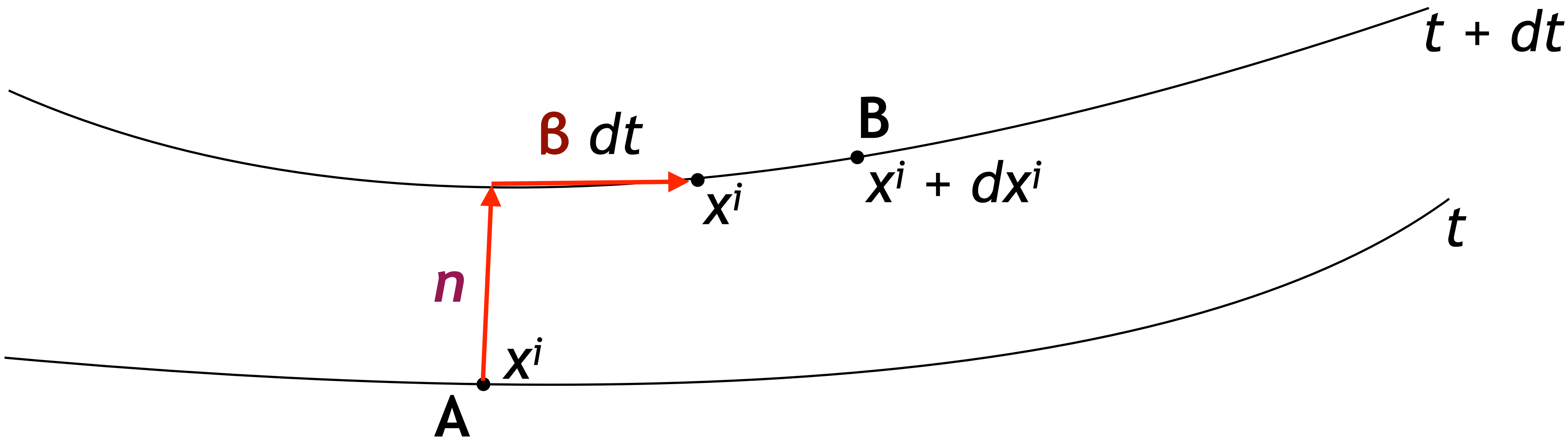

Event **A**: (t, x^i) Event **B**: $(t + dt, x^i + dx^i)$

$\beta^i \, dt$ is coordinate displacement of x^i in slice $t + dt$ from x^i in slice t. Called the "shift"; reflects freedom to slide spatial coordinates around in each timeslice.

The lapse α and the shift β^i generalize the notion of "gauge" freedom to a generic situation.

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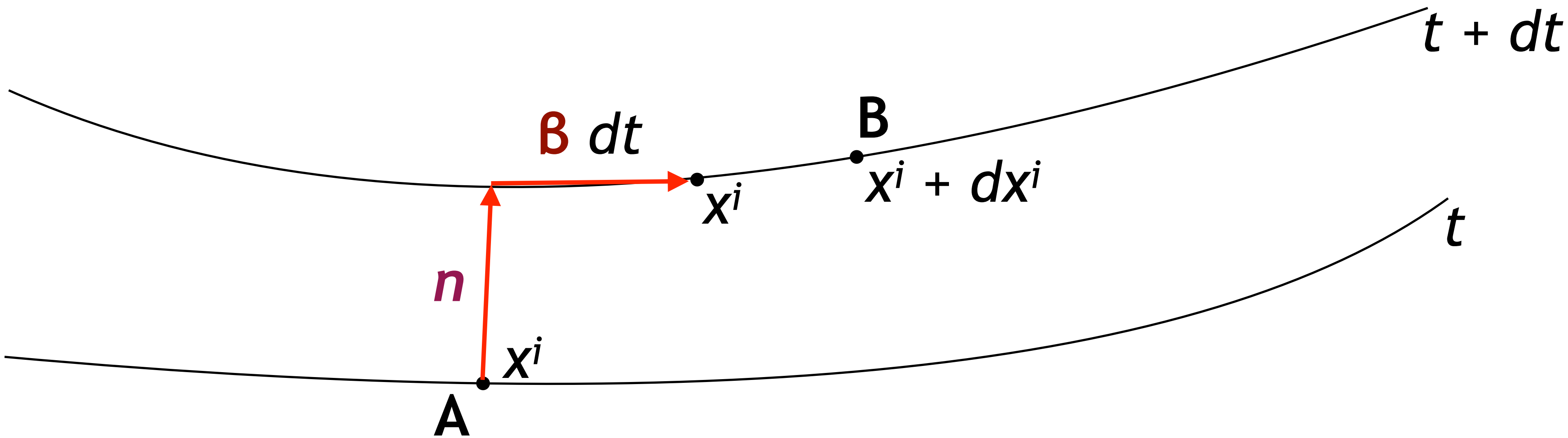


Event **A**: (t, x^i) Event **B**: $(t + dt, x^i + dx^i)$

Total spacetime distance between events **A** and **B**:

$$ds^2 = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i\, dt)(dx^j + \beta^j\, dt)$$

Some more careful definitions

Take spacetime manifold, "foliate" it with level surfaces of some scalar function *t*. Define a 1-form

$$\Omega_a = \nabla_a t \qquad \text{with norm} \qquad g^{ab}\Omega_a\Omega_b = -\frac{1}{\alpha^2}$$

Normalize this: $$\omega_a = \alpha\Omega_a$$

Define the corresponding vector: $$n^a = -g^{ab}\omega_a$$

n^a is the future-directed normal to the level surface of constant *t*. Not hard to show that $n^a n_a = -1$, can be regarded as the 4-velocity of a particular observer.

Auxiliary definition: $$t^a = \alpha n^a + \beta^a$$

β^a gives gauge freedom: can slide spatial coordinates around on each slice as we wish or need.

Some more careful definitions

Using this, define tensor that projects orthogonal to n^a:

$$\gamma_{ab} = g_{ab} + n_a n_b$$

This is tensor describes space geometry in the constant *t* "slice" ... it *is* the metric for the slice's 3-geometry.

Any tensor in a slice is then given by contracting:

$$[A^a{}_b]_{\text{in slice}} = \gamma^a{}_c \gamma^b{}_d A^c{}_d$$

Particularly useful: covariant derivative in slice:

$$\left[D_a A^b\right]_{\text{in slice}} = \gamma^c{}_a \gamma^b{}_d \nabla_c A^d$$

Can show that $D_a \gamma_{bc} = 0$ ... allows us to define Christoffel symbols in slice, write usual covariant derivative formula in any time slice.

Curvature

Last thing we need to do is develop the curvature of spacetime in this language. Two pieces:

1. *Intrinsic*: The curvature in a particular time slice. Just use γ_{ab}, develop Riemann as usual.

2. *Extrinsic*: Curvature due to how each time slice is "embedded" in the 4-dimensional geometry.

This last notion of curvature is related to the expansion or divergence of the normal vectors.

Define: Expansion $\theta_{ab} = \gamma^{c}{}_{a}\gamma^{d}{}_{b}\nabla_{c}n_{d}$

Flip sign to be in accord with usual notions of curvature

The "extrinsic" curvature is then defined as

$$K_{ab} = -\gamma^c{}_a \gamma^d{}_b \nabla_c n_d$$

With some manipulation, can show that this is simply related to the *Lie derivative* of the spatial metric:

$$K_{ab} = -\frac{1}{2}\mathcal{L}_{\vec{n}}\gamma_{ab}$$

Now, “just” need to project the Einstein field equations.

1. Time-time piece.

$$n^a n^b \, {}^{(4)}G_{ab} = 8\pi G T_{ab} n^a n^b \equiv 8\pi G \rho$$

This becomes

$$R + K^2 - K_{ab}K^{ab} = 16\pi G \rho$$

Known as the “Hamiltonian constraint.” Relates geometry in a particular slice to the energy density in that slice as measured by the observer whose 4-velocity is n^a.

Now, "just" need to project the Einstein field equations.

2. Time-space piece.

$$\gamma^a{}_c n^b G_{ab} = 8\pi G T_{ab} \gamma^a{}_c n^b \equiv -8\pi G j_c$$

This becomes

$$D_b K^b{}_a - D_a K = 8\pi G j_a$$

Known as the "Momentum constraint." Relates geometry in a particular slice to the momentum density in that slice as measured by the observer whose 4-velocity is n^a.

Now, “just” need to project the Einstein field equations.

3. Space-space piece.

$$\gamma^a{}_c \gamma^b{}_d G_{ab} = 8\pi G T_{ab} \gamma^a{}_c \gamma^b{}_d \equiv 8\pi G S_{cd}$$

This becomes

$$\mathcal{L}_{\vec{t}} K_{ab} = -D_a D_b \alpha + \alpha(R_{ab} - 2K_{ac}K^c{}_b + K K_{ab})$$
$$- 8\pi G\alpha \left[S_{ab} - \frac{1}{2}\gamma_{ab}(S - \rho)\right] + \mathcal{L}_{\vec{\beta}} K_{ab}$$

Known as the “Evolution equation.” Tells us how geometry evolves from time slice to time slice.

Theorem: Start with a slice that satisfies constraints; evolve; slice will continue to satisfy constraints.

Analogy to Maxwell: Constraints similar to divergence equations; evolution is similar to curl equations.

$$\nabla \cdot \mathbf{E} = 4\pi\rho_Q \qquad \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - 4\pi\mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

If (***E***, ***B***) obey divergence equations at an initial time, and you evolve forward in time with curl equations, then (***E***, ***B***) satisfy divergence equations at any later time.

Recipe

1. Pick coordinates. Amounts to coming up with a way of choosing lapse α and shift β^a.

2. Pick initial spacetime geometry. **Highly** nontrivial: Need to make γ_{ab} and K_{ab} that describe the situation you want to study, subject to the Hamiltonian and momentum constraints. Example: Two objects in a binary orbit. Might want to imagine they are enough apart that the post-Newtonian expansion describes them.

3. Evolve. ***If*** all is set up correctly, GR should just do its thing. For example, it should "automagically" respect ingoing boundary condition at event horizons. (Whose locations we cannot know in a dynamical spacetime until the entire calculation is completed.)

Typical result for several decades: **Catastrophic failure.**

Reason: “Constraint violating modes”

Initial data (by construction) satisfies constraints … up to a certain level of precision! Numerical noise/roundoff error will introduce “pollution” that violates constraints.

Thanks to nonlinear character of evolution equation, this “pollution” will often *grow* as spacetime evolves. Get solution dominated by nonphysical data — **not** a valid spacetime.

Brief history

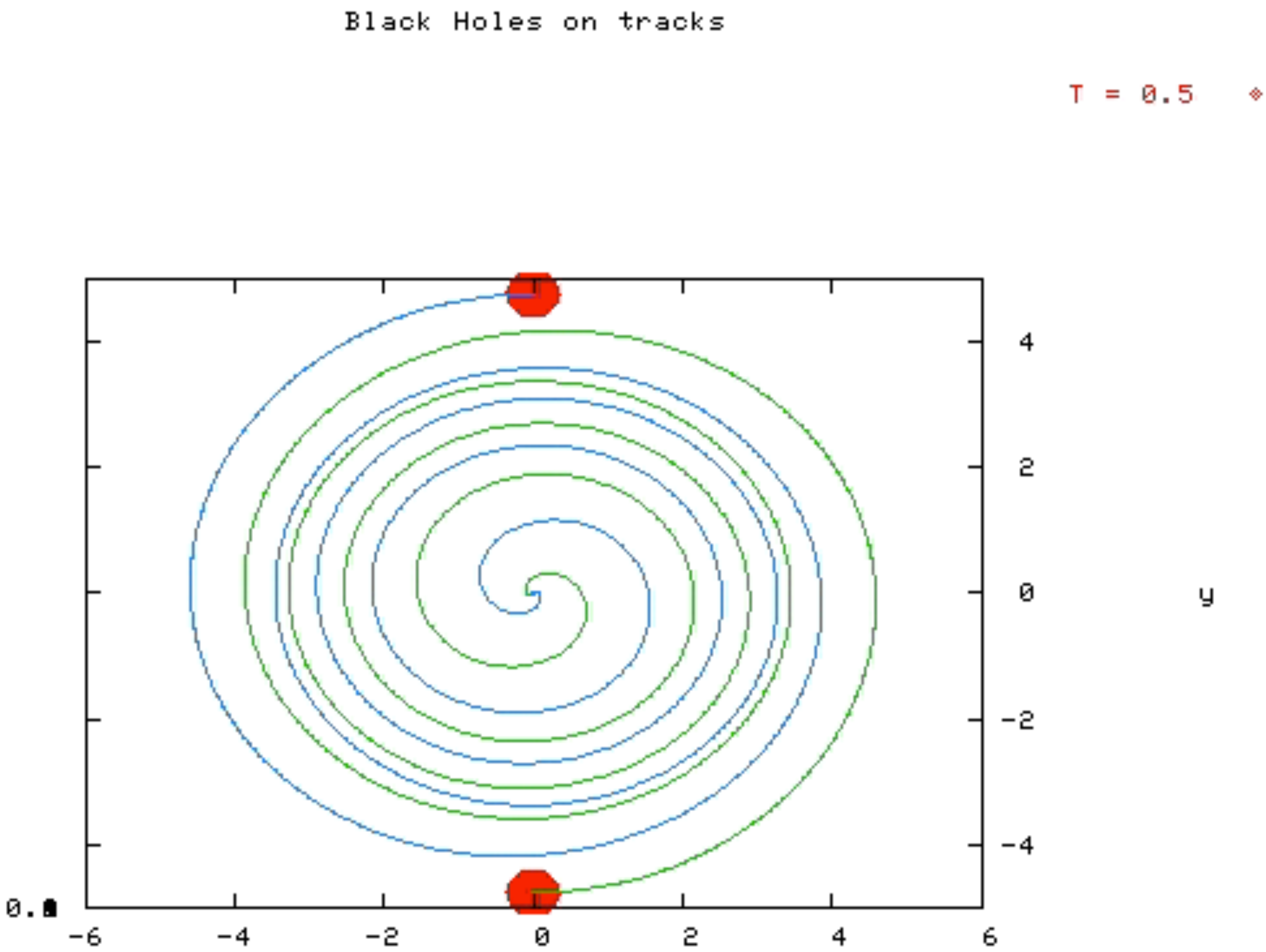
First numerical solutions attempted in 1970s for highly symmetric situations. By 1990s, people were good at doing "2+1" problems (2 space, 1 time — e.g., axial symmetry). Full 3-D was proving challenging.

Example: Anninos et al 1995 (arXiv:gr-qc/9503025). Initial data describing a single, spherical black hole. evolves with no assumptions about symmetry.

Code runs until $t \sim 50GM$ then crashes!

First attempt at evolving initial data that looked like orbiting black holes: Bernd Brügmann 1997 (arXiv:gr-qc/9708035). Black holes orbited for about $10GM$ before the code crash (roughly 1/4 of an orbit).

First black hole evolutions



Has now become routine!