Light propagating in the Schwarzschild spacetime

In lecture on Tuesday May 7th, I used the behavior of light propagating in Schwarzschild to motivate the fact that apparent pathologies in this spacetime can be traced back to the poor behavior of the coordinate t as we approach $r = 2GM$. In the course of this analysis, Enrique Mendez asked a very good question that boiled down to how an observer at some r that is *not* far from $2GM$ would define the energy of the light pulse. Because this was a packed lecture, I unfortunately was not able to deal with his question in that moment. On reflection later that day, I thought up an answer which I believe helps clarify the situation.

Begin by noting that a static observer at some r has a 4-velocity $\vec{u}$ with components

$$u^\alpha \doteq \left(\frac{dt}{d\tau_r}, \frac{dr}{d\tau_r}, \frac{d\theta}{d\tau_r}, \frac{d\phi}{d\tau_r}\right) = \left[\left(1 - \frac{2GM}{r}\right)^{-1/2}, 0, 0, 0\right]$$

The notation $d\tau_r$ is a reminder that this is an interval of proper time *as measured at* r. Note that the observer with this $\vec{u}$ is *not* a freely-falling observer: some mechanism or agent is holding this person fixed here. Note also that $dt/d\tau_r$ diverges at $r = 2GM$, and becomes imaginary for $r < 2GM$ — a reflection of pathologies in the coordinate t at this radius.

The meaning of $dt/d\tau_r$ is very important here. Bearing in mind that t is equal to the proper time of observers very far away ($r \gg 2GM$), this component of the 4-velocity tells the relative ticking rate of clocks at r with very distant clocks. *There is no constraint on the nature of these clocks.* The clock could be your wristwatch; it could be your heartbeat; it could be the frequency associated with a beam of light.

With this in mind, let's forget about energy and just focus on the ticking of clocks. Take our clock to be the frequency of a beam of light[1]. Suppose that we sit at $r_{\rm emit}$ and point a beam of light outward. Let us suppose it is a green laserpointer, with wavelength $\lambda = 532$ nanometers. *According to the observers sitting at* $r_{\rm emit}$, the electromagnetic field oscillates with a period $\Delta\tau_{\rm emit} = \lambda/c = 1.77455 \times 10^{-15}$ seconds.

This light propagates outward. Imagine it propagates past an array of static observers, each of which measures the light using their own clock. Their clocks tick with different time standards; but, using the properties of Schwarzschild, we can easily compare the different clocks. Using the fact that $u^t(r) = dt/d\tau_r$, we see that the period of the light as measured at radius r is given by

$$\Delta\tau_r = \Delta\tau_{\rm emit}\left[\frac{1 - 2GM/r}{1 - 2GM/r_{\rm emit}}\right]^{1/2} .$$

Considering observers who are very far away, this limits to

$$\Delta\tau_\infty = \Delta\tau_{\rm emit}\left(1 - \frac{2GM}{r_{\rm emit}}\right)^{-1/2} ;$$

or, using $\lambda_r = c\Delta\tau_r$, the wavelength we measure far away is

$$\lambda_\infty = \lambda_{\rm emit}\left(1 - \frac{2GM}{r_{\rm emit}}\right)^{-1/2} .$$

If our emit our beam of light at $r_{\rm emit} = 6GM$, then our 532 nanometer light beam has wavelength 652 nanometers far away (red or reddish orange). For $r_{\rm emit} = 4GM$, we get 752 nanometers (deep red to near infrared). For $r_{\rm emit} = 3GM$, we get 921 nanometers (near infrared to infrared). As we get closer and closer to $r_{\rm emit} = 2GM$, the wavelength just gets longer and longer, getting quite

[1] This is what LIGO does, using a laser to provide a precise frequency standard for examining how light travel times vary.

steep as we get close. Writing $r_{\rm emit} = (2+\epsilon)M$, the light has wavelengths microns when $\epsilon \simeq 0.1$; it is millimeters for $\epsilon \sim 10^{-6}$; it is meters for $\epsilon \sim 10^{-12}$. The wavelength diverges as $r_{\rm emit} \to 2GM$.

This calculation also illustrates why if we emit pulses with a spacing $\delta\tau_{\rm emit}$, the interval between pulses diverges. Suppose we flash out laser once every second at $r_{\rm emit}$. If $r_{\rm emit} = (2+\epsilon)M$, then distant observers see a pulse every 4.6 seconds for $\epsilon = 0.1$; every 14 seconds for $\epsilon = 0.01$; every 45 seconds for $\epsilon = 0.001$; every 20 minutes for $\epsilon = 10^{-6}$; every two weeks for $\epsilon = 10^{-12}$.

As our emitters approach $r = 2GM$, the light by which we observe them becomes infinitely dilated, diverging the light's wavelength to a scale where we cannot observe it, and inflating any interval between pulses to longer and longer scales.

Let's now consider energy and this beam of light. As I pondered Enrique's question, I realized that part of the confusion is that there are two notions of energy that are being used, and it is important to distinguish between them:

- The *conserved energy*, $E_{\rm cons} = -p_t$, which arises because the Schwarzschild spacetime is time independent and hence possesses a timelike Killing vector; and
- The *measured energy*, $E_{\vec u} = -p_\alpha u^\alpha$, which is the energy as measured by an observer whose 4-velocity is $\vec u$.

These two notions of energy *are not the same in general.* They describe different things: $E_{\rm cons}$ is an invariant of the motion that arises in much the same way that a conserved energy arises from a time-independent Lagrangian in non-relativistic physics; $E_{\vec u}$ is what the observer with 4-velocity $\vec u$ would find if they probe the light using some kind of measurement device.

Let us use the 4-velocity for a static observer, $\vec u$ as given above. In this case, since the only non-zero 4-velocity component is u^t,

$$E_{\vec u} = -p_t u^t = E_{\rm cons}\left(1 - \frac{2M}{r}\right)^{-1/2} .$$

Notice that $E_{\vec u} \to E_{\rm cons}$ as $r \to \infty$. Let's call this E_∞, the energy that is measured by a very distant observer. Let us label as $E_{\rm meas}(r)$ the energy that is measured by the observer with $\vec u$. Then, manipulating this expression, we see that

$$E_\infty = E_{\rm meas}(r)\left(1 - \frac{2GM}{r}\right)^{1/2} .$$

So a pulse of radiation that is emitted at r with energy $E_{\rm meas}(r)$ has energy E_∞ by the time that it gets very far away. This E_∞ is less than $E_{\rm meas}(r)$; in fact, $E_\infty \to 0$ as $r \to 2GM$.

Hopefully you can see that the discussion of the light's energy is consistent with the previous discussion of how its wavelength dilates as it propagates from r near $2GM$ to very distant observers.