Applying the first law of thermodynamics to a polytrope

Today in lecture, I described how to use the first law of thermodynamics, \( dU = -P \, dV \) in concert with a polytrope pressure law \( P = K \rho_0^\Gamma \) expressed in terms of rest energy density to make a simple relationship between rest energy density and energy density \( \rho \). My lecture notes elided a bit more of the details than I liked; in checking them over, I’ve actually found a cleaner way to get the final result, which is described here.

Key to this trick is to imagine a fiducial volume \( V \) which contains rest energy \( m_0 \) and energy \( m \). Then, \( \rho_0 = m_0/V, \, \rho = m/V \). In terms of these variables, the fiducial volume is \( V = (1/m_0)(1/\rho_0) \), and the energy in that fiducial volume is \( U = \rho V = (1/m_0)(\rho/\rho_0) \).

In terms of these things, the first law becomes

\[
d \left( \frac{\rho}{\rho_0} \right) = -P \, d \left( \frac{1}{\rho_0} \right)
\]

Expanding the differentials we have

\[
\frac{d\rho}{\rho_0} - \frac{\rho \, d\rho_0}{\rho_0^2} = \frac{P}{\rho_0^2} d\rho_0
\]

which rearranges to

\[
d\rho = \left( \frac{\rho_0 + P}{\rho_0} \right) d\rho_0 = (1 + K \rho_0^{\Gamma-1}) d\rho_0
\]

This immediately integrates up to

\[
\rho = \rho_0 + \frac{K \rho_0^\Gamma}{\Gamma - 1} = \rho_0 + \frac{P}{\Gamma - 1}
\]

Notice that this has the correct limiting behavior \( \rho \to \rho_0 \) as \( P \to 0 \).