Applying the first law of thermodynamics to a polytrope

In lecture, I described how to use the first law of thermodynamics, $dU = -P\,dV$ in concert with a polytrope pressure law $P = K\rho_0^\Gamma$ expressed in terms of *rest*-energy density to make a simple relationship between rest-energy density and energy density ρ. My lecture notes elided more of the details than I liked. These notes expand these details; they also supersede a set of notes I had posted earlier, which were flawed.

Key to this trick is to imagine a fiducial volume V which contains rest energy m_0 and energy m. Then, $\rho_0 = m_0/V$, $\rho = m/V$. In terms of these variables, the fiducial volume is $V = (1/m_0)(1/\rho_0)$, and the energy in that fiducial volume is $U = \rho V = (1/m_0)(\rho/\rho_0)$.

In terms of these things, the first law becomes

$$d\left(\frac{\rho}{\rho_0}\right) = -P\,d\left(\frac{1}{\rho_0}\right)$$

On the left-hand side, let us write $\rho_0 = (P/K)^{1/\Gamma}$, so that we have

$$\begin{aligned} d\left(\frac{\rho}{\rho_0}\right) &= -PK^{1/\Gamma}d\left(P^{-1/\Gamma}\right) = \frac{K^{1/\Gamma}}{\Gamma}\frac{P\,dP}{P^{1+1/\Gamma}} \\ &= \frac{K^{1/\Gamma}}{\Gamma}\frac{dP}{P^{1/\Gamma}} \end{aligned}$$

Now, integrate both sides over over pressure, from a lower limit of 0 up to P. Here, we impose a boundary condition: at very low pressure, the rest-energy density and the energy density are the same: $\rho \to \rho_0$ as $P \to 0$. The result is

$$\begin{aligned} \frac{\rho}{\rho_0} - 1 &= \frac{K^{1/\Gamma}}{\Gamma}\frac{\Gamma\,P^{1-1/\Gamma}}{\Gamma - 1} = \frac{P}{\Gamma - 1}\left(\frac{K}{P}\right)^{1/\Gamma} \\ &= \frac{P}{\Gamma - 1}\left(\frac{1}{\rho_0}\right) \end{aligned}$$

from which we immediately see that

$$\rho = \rho_0 + \frac{P}{\Gamma - 1}$$

This is the result quoted in the lecture notes.