Gauge transformations on a curved background

This notes accompany the discussion of the “generalized gauge transformation” given in Lecture 17, page 4. Begin with a spacetime whose metric we represent as some background $\hat{g}_{\alpha\beta}$ plus a perturbation $h_{\alpha\beta}$; these are all expressed in terms of the coordinate $x^\gamma$. We wish to change to a new coordinate $y^\gamma$ which is related to $x^\gamma$ by an infinitesimal shift $\xi^\gamma$:

$$x^\alpha = y^\alpha - \xi^\alpha(y^\gamma).$$

This coordinate transformation changes the metric according to

$$g_{\alpha\beta}(y^\gamma) = \hat{g}_{\alpha\beta}(x^\gamma) \frac{\partial x^\gamma}{\partial y^\alpha} \frac{\partial x^\delta}{\partial y^\beta}.$$

Let’s now expand this out:

$$g_{\alpha\beta}(y^\gamma) = (\hat{g}_{\gamma\delta} + h_{\gamma\delta}) (\delta^\gamma_\alpha - \partial_\alpha \xi^\gamma) \left( \delta^\delta_\beta - \partial_\beta \xi^\delta \right).$$

On going from the fourth to the fifth line, we’ve used the definition of the Christoffel symbol. On that last line, the background in this expression is currently expressed as a function of the old coordinates, $x^\gamma$. We want it to appear as a function of the new coordinates, $y^\gamma$. We want it to appear as a function of the new coordinates, $y^\gamma$:

$$\hat{g}_{\alpha\beta}(x^\gamma) = \hat{g}_{\alpha\beta}(y^\gamma - \xi^\gamma).$$

Note, we could do this expansion for the perturbation as well: $h_{\alpha\beta}(x^\gamma) = h_{\alpha\beta}(y^\gamma - \xi^\gamma) = h_{\alpha\beta}(y^\gamma) - \xi^\gamma \partial_\gamma h_{\alpha\beta}$. This means that $h_{\alpha\beta}(x^\gamma) = h_{\alpha\beta}(y^\gamma)$+ terms that are of order “small squared,” which we neglect.

Substituting the result for $\hat{g}_{\alpha\beta}(x^\gamma)$ into $g_{\alpha\beta}(y^\gamma)$ yields

$$g_{\alpha\beta}(y^\gamma) = \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{\nabla}_a \xi_\beta - \hat{\nabla}_\beta \xi_a,$$

from which we read off

$$h_{\alpha\beta} \rightarrow h_{\alpha\beta} - \hat{\nabla}_a \xi_\beta - \hat{\nabla}_\beta \xi_a.$$