MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.962 SPRING 2024

LECTURE 1
SPACETIME AND GEOMETRY PART I

1.1 Goal of this course

General relativity is Albert Einstein's relativistically covariant theory of gravity. It famously makes gravity a consequence of the geometry of spacetime. The goal of the version of 8.962 taught by Scott Hughes is for you to learn the fundamentals of general relativity (roughly the first half, up to spring break) and to explore some of its most important consequences, particularly as they pertain to our universe as we observe it today, and some of the most interesting and important gravitating objects that we encounter (the second half, from spring break to the end of term).

We will assume in this course that you are familiar with spacetime as described by special relativity. Special relativity turns out to have geometric notions embedded in it, but it is possible to do a lot in special relativity without explicitly thinking about geometry. However, to go from special relativity to general relativity, we need to understand the geometry. So the start of 8.962 will largely consist of a review and re-presentation of special relativity using mathematical language that describes it geometrically, doing so in a way that carries over naturally to general relativity. The fundamental observation is that special relativity is the framework in which measurements and observations are subject to Lorentz symmetry, and thus measurements and observations made by different inertial observers can be related by Lorentz transformations.

1.2 Spacetime: Critical definitions and properties

Spacetime is a *manifold* of *events* that is endowed with a *metric*. Several terms in this opening sentence require definitions:

- *Manifold*: For our purposes, a manifold is a set of points with well-understood connectedness properties (i.e., with a defined topology). We require that it have certain well-defined, "nice" smoothness properties, and that the neighborhood of any point should look like Euclidean space (or like Lorentzian space if it includes timelike directions). See Carroll pages 54–62 (plus additional discussion in Appendix A) for very careful and rigorous discussion. The mathematical machinery that Carroll introduces goes beyond what we need in 8.962, but it is very much worth it for you to be familiar with this terminology (and, depending on your interests, you might find that this level of rigor is useful for you). Wald's textbook, chapter 2 (particularly section 2.1), is also likely to be of interest to students who desire a more mathematical formulation.

- *Event*: When and where something happens. We label events with coordinates, but the event itself is a geometric object whose meaning is independent of the coordinate representation. For example, if "event A" corresponds to a piece of chalk hitting the floor at a moment during lecture, I might assign this event the coordinates $(t, x, y, z) =$ (Feb 6 2:37 pm, 0, 0, 0); a student in the front row might assign it the coordinates (Feb 6 2:37pm, 2 meters, 0, -0.5 meters); a student in the back row whose watch is set for a different time zone might assign it the coordinates (Feb 6 7:37pm, 4 meters, 1 meter, -1 meter). We all use different coordinates to represent this event, but our different representations nonetheless describe the same point in spacetime. Many of the most important concepts and structures we will develop in this class can be thought of as "geometric objects": entities describing some aspect of physics whose nature is independent of coordinates. Although the physical meaning of these objects "transcend" their coordinate description, we will nonetheless use coordinates for many of our calculations, and so it will be important to understand how to convert between different observers' various coordinates.

- *Metric*: A notion of distance between events in spacetime. This comes from the physics attached to these events. A manifold on its own does not include a notion of distance between its constituent points; we will develop the concept of metric in more detail in the coming discussion. Key for us is that the metric is what puts geometry into the manifold.

SR is essentially the simplest theory of spacetime consistent with most of the physics that we observe; it turns out to correspond to the zero gravity limit of general relativity (GR). A key concept for us in this discussion is the notion of an *inertial reference frame*, or IRF. Following Blandford and Thorne, we carefully define this frame as *a (conceptual) lattice of clocks and measuring rods that allows us to assign coordinates to (i.e., to label) spacetime events.* The IRF and this lattice have the following properties:

- The lattice moves freely through spacetime — no forces act on it, and it does not rotate relative to distant beacons.
- The measuring rods are orthogonal and uniformly ticked, forming an orthonormal, Cartesian coordinate system.
- All of the clocks tick uniformly.
- The clocks are synchronized using the "Einstein synchronization procedure": clock 1 emits a pulse of light at $t = t_e$. It bounces off a mirror on clock 2, and is received back at clock 1 at $t = t_r$. Clock 2 is synchronized with clock 1 such that the time of bounce is $t_b = (t_e + t_r)/2$. This synchronization is done between every pair of clocks in the lattice.

⌈Notice that the Einstein synchronization procedure takes advantage of the fact that the speed of light c is a relativistic invariant: it is a "good" synchronization procedure because the light travel time between various events (emission of light, bouncing of light, reception of light) depends upon a quantity whose value is agreed upon by all observers. Never forget the important role that the propagation of light has in understanding the way that time is measured! Later in the course, we will encounter some situations in which time seems to be behaving oddly, at least according to certain observers. The reason for this is going to be intimately tied up with the way that light propagates through spacetime.⌋

An event is represented in this IRF by the coordinates: $\mathcal{P} \doteq (t_{\mathcal{P}}, x_{\mathcal{P}}, y_{\mathcal{P}}, z_{\mathcal{P}})$. (Throughout these notes, I use the symbol "$\underset{\mathcal{O}}{\doteq}$" to mean "the object on the left-hand side is represented by the set on the right-hand side in the IRF of observer $\mathcal{O}$." If there is no ambiguity about or need to carefully specify the observer, I may simply write this "$\doteq$".) This is a good moment for a brief aside on the system of units we use: If our clocks tick once per second, then we choose our tickmarks such that the basic unit of length is 1 light second. In these units, the speed of light is then

$$c = \frac{1\,\text{lightsecond}}{\text{second}} \equiv 1\ . \tag{1.1}$$

In this system, the speed of light $c = 1$, and all speeds are dimensionless. To convert to "normal" units, think of all speeds and velocities as being scaled to c: $v = v_{\rm normal}/c$.

Let $\mathcal{O}$ be an observer who uses this IRF. We can now define a *displacement vector* between 2 spacetime events, $\mathcal{Q}$ and $\mathcal{P}$:

$$\begin{aligned} \Delta\vec{x} &\underset{\mathcal{O}}{\doteq} (t_{\mathcal{Q}} - t_{\mathcal{P}}, x_{\mathcal{Q}} - x_{\mathcal{P}}, y_{\mathcal{Q}} - y_{\mathcal{P}}, z_{\mathcal{Q}} - z_{\mathcal{P}}) \\ &\underset{\mathcal{O}}{\doteq} (\Delta t, \Delta x, \Delta y, \Delta z) \ . \end{aligned} \tag{1.2}$$

We will denote the components of such a "4-vector" (so-called because it has 4 components, one for each spacetime direction) as Δx^μ (or using another letter from the Greek alphabet, depending on the needs of the calculation). When Greek letters are used to label an index, they are taken from the set $[0, 1, 2, 3]$, with direction 0 being along the "time" axis in that frame, and indices 1, 2, and 3 the space indices (typically ordered in a right-hand fashion, e.g., $1 \to x$, $2 \to y$, $3 \to z$ in a Cartesian coordinate system). We will sometimes only want the spatial pieces (at least, the spatial pieces according to some specified observer):

$$\Delta\mathbf{x} = \Delta\underset{\sim}{x} \underset{\mathcal{O}}{\doteq} (x_{\mathcal{Q}} - x_{\mathcal{P}}, y_{\mathcal{Q}} - y_{\mathcal{P}}, z_{\mathcal{Q}} - z_{\mathcal{P}}) \ . \tag{1.3}$$

We will tend to use the boldface notation for spatial quantities that we typeset, and the undertilde when writing by hand. Components of a spatial vector are denoted using lowercase Latin letters, and range over 1, 2, 3. So, in a right-handed Cartesian coordinate system, Δx^i is shorthand for $(\Delta x^1, \Delta x^2, \Delta x^3) = (\Delta x, \Delta y, \Delta z)$.

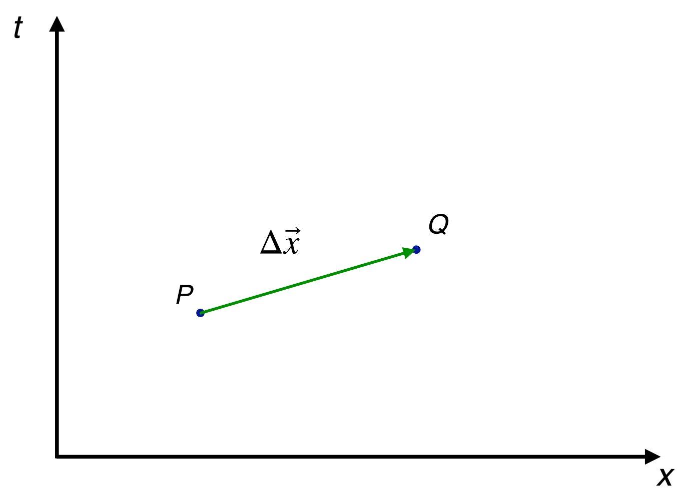

Figure 1: The displacement $\Delta\vec{x}$ between two spacetime events $\mathcal{P}$ and $\mathcal{Q}$, along with coordinate axis used by an observer in some IRF.

1.3 Lorentz transformations

We have formulated everything so far in terms of measurements made by a particular observer $\mathcal{O}$. Let us consider the events $\mathcal{P}$ and $\mathcal{Q}$ and the displacement vector $\Delta\vec{x}$ that connects them, but now as viewed by an observer $\bar{\mathcal{O}}$. The key principle we bear in mind is that $\mathcal{P}$, $\mathcal{Q}$, and $\Delta\vec{x}$ *do not change*, since they are simply geometric objects in the manifold of spacetime. However, *their representation according to $\bar{\mathcal{O}}$ is not the same as their representation according to $\mathcal{O}$.*

Because we take our starting point to be the special theory of relativity, the representations used by these two observers are related by the Lorentz transformation. Let us suppose that the IRF of $\bar{\mathcal{O}}$ moves with speed v along the "1" (i.e., x) direction as seen by IRF $\mathcal{O}$. Then, denoting

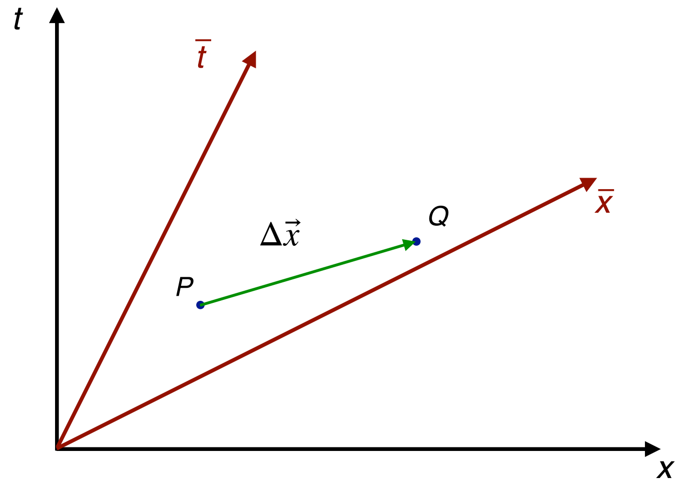

Figure 2: The displacement $\Delta\vec{x}$ between two spacetime events $\mathcal{P}$ and $\mathcal{Q}$, along with coordinate axis used by two observers in two different IRFs. The events and the displacement are the same geometric objects, though the two observers label these objects differently.

by $x^{\bar{\mu}}$ the components in $\bar{\mathcal{O}}$'s representation, we have

$$\begin{aligned} \Delta x^{\bar{0}} &= \gamma\Delta x^0 - \gamma v\Delta x^1 & (1.4)\\ \Delta x^{\bar{1}} &= -\gamma v\Delta x^0 + \gamma\Delta x^1 & (1.5)\\ \Delta x^{\bar{2}} &= \Delta x^2 & (1.6)\\ \Delta x^{\bar{3}} &= \Delta x^3 \; . & (1.7) \end{aligned}$$

This can be written more compactly

$$\Delta x^{\bar{\mu}} = \sum_{\nu=0}^{3} \Lambda^{\bar{\mu}}{}_{\nu}\Delta x^{\nu} \equiv \Lambda^{\bar{\mu}}{}_{\nu}\Delta x^{\nu} \; . \tag{1.8}$$

Here, we have introduced the Lorentz transformation matrix,

$$\Lambda^{\bar{\mu}}{}_{\nu} \doteq \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{1.9}$$

as well as the Einstein summation convention: indices on the same side of the equal sign that are repeated, with one in the "upstairs" position and the other "downstairs," are taken to be summed from 0 to 3. The quantity $\gamma = 1/\sqrt{1-v^2}$.

Notice that one can write

$$\Lambda^{\bar{\mu}}{}_{\nu} = \frac{\partial x^{\bar{\mu}}}{\partial x^{\nu}} \; . \tag{1.10}$$

The right-hand side of Eq. (1.10) can be regarded as defining a more general notion of transforming between different representations of the spacetime displacement, with the Lorentz transformation as a particularly important special case.

In Eq. (1.8), it is irrelevant what label we use for the index corresponding to $\mathcal{O}$'s coordinates, as long as this label is distinct from all other index labels in the problem. Because we sum over the full range of the index being summed upon,

$$\Delta x^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\nu}x^{\nu} = \Lambda^{\bar{\mu}}{}_{\alpha}x^{\alpha} = \Lambda^{\bar{\mu}}{}_{\beta}x^{\beta} = \ldots \tag{1.11}$$

are all equivalent and equally good ways of representing the components $\Delta x^{\bar{\mu}}$ used by $\bar{\mathcal{O}}$ to the components used by $\mathcal{O}$. The index which is summed over is known as a *dummy index*. We will occasionally exploit the fact that the particular label we use for it is irrelevant to re-organize and simplify certain calculations which involve multiple index sums. On the other hand, we do not have freedom to relabel the index $\bar{\mu}$, which is not summed over and appears on both sides of the equation. It is often called a "free index."

1.4 Vectors more generally

The spacetime displacement $\Delta\vec{x} \doteq (\Delta t, \Delta x, \Delta y, \Delta z)$ is our prototype for a vector in spacetime, or 4-vector. We will generalize the notion by defining any quartet of numbers — "components" — that transforms between IRFs like the displacement vector to be a vector:

$$\vec{A} \underset{\mathcal{O}}{\doteq} (A^0, A^1, A^2, A^3) , \tag{1.12}$$

$$A^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_{\alpha} A^{\alpha} . \tag{1.13}$$

That this quartet of numbers is subject to this transformation law is what makes them vector components, and not just some set of numbers. We also require that these components obey the linearity laws which go into a vector space:

- If $\vec{A}$ is a vector (with components A^α) and a is a scalar[1], then $a\vec{A}$ is a vector (with components aA^α).
- If $\vec{A}$ and $\vec{B}$ are both vectors (with components A^α and B^α, respectively), then $\vec{C} = \vec{A} + \vec{B}$ is a vector (with components $C^\alpha = A^\alpha + B^\alpha$).

Within frame $\mathcal{O}$, we can define 4 clearly very special vectors:

$$\vec{e}_0 \doteq (1,0,0,0) , \qquad \vec{e}_1 \doteq (0,1,0,0) ,$$
$$\vec{e}_2 \doteq (0,0,1,0) , \qquad \vec{e}_3 \doteq (0,0,0,1) . \tag{1.14}$$

We call these *basis vectors*, since they pick out 4 principle directions associated with the coordinates used in $\mathcal{O}$. (Note that we cannot call the basis vectors "unit" vectors, at least not yet, since we have not ascertained how to compute a "magnitude" associated with each one. Hold that thought for another page or so.) Note that the index on each vector is not a component; instead, it labels which member of the set is being referenced. The components of the basis vectors can be written compactly using the Kronecker delta:

$$(\vec{e}_\alpha)^\beta = \delta_\alpha{}^\beta . \tag{1.15}$$

Using the basis vectors, we can write a vector in terms of its components:

$$\vec{A} = A^\alpha \vec{e}_\alpha . \tag{1.16}$$

Notice that this is a "real" equality, not the $\doteq$ that we use to indicate that a representation holds in some specified reference frame. This means that the equality holds in all IRFs, a fact we can exploit to deduce how the basis vectors transform between different IRFs:

$$\begin{aligned}
\vec{A} = A^\alpha \vec{e}_\alpha &= A^{\bar{\mu}} \vec{e}_{\bar{\mu}} \quad \text{(Equality holds in all IRFs)} \\
&= \left(\Lambda^{\bar{\mu}}{}_{\beta} A^\beta\right) \vec{e}_{\bar{\mu}} \quad \text{(Lorentz transformation of components)} \\
&= \left(A^\beta \Lambda^{\bar{\mu}}{}_{\beta}\right) \vec{e}_{\bar{\mu}} \quad \text{(Order does not matter in index notation)} \\
&= \left(A^\alpha \Lambda^{\bar{\mu}}{}_{\alpha}\right) \vec{e}_{\bar{\mu}} \quad \text{(Relabel dummy indices)}
\end{aligned} \tag{1.17}$$

[1] Note, we haven't actually discussed what a scalar is! Hold that thought for a page or two.

With this we rearrange the equation as follows:

$$A^\alpha \left(\vec{e}_\alpha - \Lambda^{\bar{\mu}}{}_\alpha \vec{e}_{\bar{\mu}} \right) = 0 \;, \tag{1.18}$$

from which we see that the rule for transforming basis vectors is given by

$$\boxed{\boxed{\vec{e}_\alpha = \Lambda^{\bar{\mu}}{}_\alpha \vec{e}_{\bar{\mu}}}} \tag{1.19}$$

Notice that this expresses the basis vectors in frame $\mathcal{O}$ in terms of the basis vectors in frame $\bar{\mathcal{O}}$ — quite different from the rule that we use to transform the components:

$$\boxed{\boxed{A^{\bar{\mu}} = \Lambda^{\bar{\mu}}{}_\alpha A^\alpha}} \tag{1.20}$$

In both cases, there's a simple algorithm: "Line up the indices." However, if your goal is to understand how to write down $\bar{\mathcal{O}}$'s basis vectors in terms of the basis vectors in $\mathcal{O}$, then we need the *inverse* Lorentz transformation.

Thanks to special relativity, it is physically clear what the inverse Lorentz transformation must be: the Lorentz transformation depends upon the relative velocity of the frames, $\mathbf{v}$, so if we want the inverse transformation, we just flip the sign of all velocity terms. I.e., if

$$\vec{e}_\alpha = \Lambda^{\bar{\mu}}{}_\alpha(\mathbf{v}) \vec{e}_{\bar{\mu}} \tag{1.21}$$

(writing the transformation as an explicit function of $\mathbf{v}$ for clarity), then

$$\vec{e}_{\bar{\mu}} = \Lambda^\alpha{}_{\bar{\mu}}(-\mathbf{v}) \vec{e}_\alpha \;. \tag{1.22}$$

The different components of $\Lambda^\alpha{}_{\bar{\mu}}(-\mathbf{v})$ are identical to the components of $\Lambda^{\bar{\mu}}{}_\alpha(\mathbf{v})$, but with the sign of $\mathbf{v}$ flipped.

To be perfectly blunt, keeping track of these different indices and the sign associated with the velocity is something of a pain in a region a bit below the lower back. My personal mnemonic is to keep in my head the matrix that we use for transforming the displacement vector's components. When we do that, the velocity which enters the components is that of the IRF associated with the top index according to observers in the IRF associated with the bottom index.

One last transformation exercise: let's combine the Lorentz transformation with its inverse.

$$\begin{aligned} \vec{e}_\alpha &= \Lambda^{\bar{\mu}}{}_\alpha(\mathbf{v}) \vec{e}_{\bar{\mu}} \\ &= \Lambda^{\bar{\mu}}{}_\alpha(\mathbf{v}) \left[\Lambda^\beta{}_{\bar{\mu}}(-\mathbf{v}) \vec{e}_\beta \right] \\ &= \left[\Lambda^{\bar{\mu}}{}_\alpha(\mathbf{v}) \Lambda^\beta{}_{\bar{\mu}}(-\mathbf{v}) \right] \vec{e}_\beta \;. \end{aligned} \tag{1.23}$$

This equality requires that

$$\boxed{\boxed{\Lambda^{\bar{\mu}}{}_\alpha \Lambda^\beta{}_{\bar{\mu}} = \delta^\beta{}_\alpha}} \tag{1.24}$$

This is simply another way of stating that these are elements of matrices which have an inverse relation to each other. One can likewise show

$$\boxed{\boxed{\Lambda^\alpha{}_{\bar{\mu}} \Lambda^{\bar{\nu}}{}_\alpha = \delta^{\bar{\nu}}{}_{\bar{\mu}}}} \tag{1.25}$$

1.5 Invariant product

Let us go back to the displacement vector $\Delta\vec{x} \doteq (\Delta t, \Delta x, \Delta y, \Delta z)$ for a moment. A very important result from special relativity[2] is that

$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 \tag{1.26}$$

is a *Lorentz invariant*: although different inertial observers will measure different values for Δt, Δx, etc., they will *all agree* on the value of Δs^2. Note that, because of this, we could use either the true equals "=" or the IRF representation symbol "$\doteq$"; the right-hand side is written using quantities that are defined in a particularly IRF, but the sum is independent of this frame. We will typically use = to emphasize the fact that all observers agree on the result, though one should remember that the terms on the right-hand vary from frame to frame.) We will define this to be a *Lorentz scalar*, or "scalar" for short: A quantity whose value is the same in all IRFs. (Incidentally, Δs^2 is often called the invariant interval.)

We will use this notion of an invariant to define a scalar or invariant product (I will use the terms interchangeably). Let us define

$$\Delta s^2 \equiv \Delta\vec{x} \cdot \Delta\vec{x}\ . \tag{1.27}$$

Another way of saying this is that

$$\Delta\vec{x} \cdot \Delta\vec{x} = -(\Delta x^0)^2 + (\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2\ . \tag{1.28}$$

This looks just like the Euclidean dot product with which you are surely familiar, but the time piece enters with a minus sign. Why? That's what physics demands in order to construct an invariant. The best I can say is that time is not on the same footing as the other dimensions: you can easily move forward and backward, left and right, up and down; but you cannot easily move to the past and the future. Physics tells the math to treat this direction differently than the others.

Since we define vectors as objects which transform like the displacement, we can develop scalar products with them in just the same way. For example,

$$\vec{A} \cdot \vec{A} = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2\ . \tag{1.29}$$

This is a good point to introduce some terminology:

- If $\vec{A} \cdot \vec{A} < 0$, we say that $\vec{A}$ is "timelike." Intuitively, the time component dominates the properties of this vector. More rigorously, you can find a Lorentz transformation into an IRF where $\vec{A}$ *only* has a timelike component — there exists a frame in which the spatial components of $\vec{A}$ are zero. (Actually, you can find infinitely many such frames: Once you have found one such frame, you can perform rotations to any number of additional frames in which $\vec{A}$ only has a timelike component.)

- If $\vec{A} \cdot \vec{A} > 0$, we say that $\vec{A}$ is "spacelike." Intuitively, the spatial components dominate this vector's properties. More rigorously, you can find Lorentz transformations such that $\vec{A}$ only has spatial components; there are frames that completely eliminate the timelike component of spacelike 4-vectors.

- If $\vec{A} \cdot \vec{A} = 0$, we say that $\vec{A}$ is "null" (meaning hopefully obvious) or "lightlike." To understand why we call this lightlike, suppose that $\vec{A} = \Delta\vec{x}$ — our 4-vector is a spacetime displacement vector. If $\Delta\vec{x} \cdot \Delta\vec{x} = 0$, then the events which define the endpoints of this displacement vector can be connected by a pulse of light. (Write it out and check if this is not clear; it may be helpful to restore factors of c.) This means that a lightlike or null vector is oriented parallel to the path that light follows in spacetime.

[2] If this looks slightly funny to you, remember that our unit system has $c = 1$.

A more general notion of the scalar product is

$$\vec{A} \cdot \vec{B} = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 \; . \tag{1.30}$$

It is simple to see that this is also invariant: consider $\vec{C} = \vec{A} + \vec{B}$. Then,

$$\vec{C} \cdot \vec{C} = \vec{A} \cdot \vec{A} + \vec{B} \cdot \vec{B} + 2\vec{A} \cdot \vec{B} \; . \tag{1.31}$$

The left-hand side and the first two terms on the right-hand side are manifestly invariant; the final term on the right-hand side therefore must be as well.

Another way of writing this final notion of scalar product is

$$\begin{aligned} \vec{A} \cdot \vec{B} &= (A^\alpha \vec{e}_\alpha) \cdot (B^\beta \vec{e}_\beta) \\ &= A^\alpha B^\beta (\vec{e}_\alpha \cdot \vec{e}_\beta) \\ &\equiv A^\alpha B^\beta \eta_{\alpha\beta} \; . \end{aligned} \tag{1.32}$$

On the last line we have defined $\vec{e}_\alpha \cdot \vec{e}_\beta \equiv \eta_{\alpha\beta}$. This is known as the *metric tensor*. Its components can be written in matrix form

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \; . \tag{1.33}$$

Strictly speaking, (1.33) should be written using $\doteq$ rather than $=$; however, this tensor turns out to be represented by these components in *all* IRFs.

Using the components of $\eta_{\alpha\beta}$, we can now assess whether the basis vectors are unit vectors:

$$\vec{e}_0 \cdot \vec{e}_0 = -1 \; , \tag{1.34}$$

$$\vec{e}_1 \cdot \vec{e}_1 = \vec{e}_2 \cdot \vec{e}_2 = \vec{e}_3 \cdot \vec{e}_3 = 1 \; , \tag{1.35}$$

$$\vec{e}_\alpha \cdot \vec{e}_\beta = 0 \quad \text{if } \alpha \neq \beta. \tag{1.36}$$

These basis vectors indeed are unit vectors, with the twist that $\vec{e}_0$ is a *timelike* unit vector (logically enough), and $\vec{e}_{1,2,3}$ are all *spacelike* unit vectors (again, logically enough). They are also orthogonal to each other (we continue to call two vectors whose scalar product is zero "orthogonal," even in spacetime), so this is in fact an orthonormal basis.