Revisit: Tensors: \( \mathfrak{T}^{\alpha}_{\beta} \) tensor is a mathematical object which linearly maps \( N \) vectors into one vector in invariant scalars.

1-forms: Special name for \( (1) \) tensors: map a single vector into scalars: \( \tilde{p} (\tilde{A}) = p^a A^a \)

1-forms live in a vector space; basis for this given by basis 1-forms \( \tilde{\omega}^a \)

Can write \( \tilde{p} = p^a \tilde{\omega}^a \)

Define basis 1-forms via \( \tilde{\omega}^a (\tilde{e}_\beta) = \delta^a_\beta \).
Example of a 4-form: consider some trajectory:

\[ \vec{u} = \frac{d\vec{x}}{dt} \text{ along traj} \]

\[ \vec{r} = (\frac{dt}{dt}, \frac{dx}{dt}) \]

\( x \): parameter along traj.

Proper time: time as measured along the trajectory.

Suppose space is filled with some scalar field \( \phi (t, x, y, z) \). What is the rate of change of \( \phi \) along this curve?

3-space Euclidean intuition:

\[ \frac{d\phi}{dt} = \frac{dx}{dt} \frac{\partial \phi}{\partial x} + \frac{dy}{dt} \frac{\partial \phi}{\partial y} + \frac{dz}{dt} \frac{\partial \phi}{\partial z} \]

\[ \approx \vec{v} \cdot \nabla \phi \]

generalize to spacetime:

\[ \frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \frac{dx}{dt} \frac{\partial \phi}{\partial x} + \ldots \]

\[ = \vec{u} \cdot \nabla \phi + u_x \frac{\partial \phi}{\partial x} + \ldots \]

\[ \frac{d\phi}{dt} = u^\alpha \frac{\partial \phi}{\partial x^\alpha} = u^\alpha \partial_{x^\alpha} \phi \]

canonical on this notation!

\( \sim \) 4-form.

Gradient is a 1-form:

\[ \nabla \phi = \partial \phi = \xi \partial x^\alpha \phi \]

Note: notation for directional derivative:

\[ \frac{d\phi}{dt} = u^\alpha \partial_{x^\alpha} \phi \approx \nabla_{\vec{u}} \phi \]

"\( x \)" and "\( y \)" and "\( z \)" denote partial derivatives with respect to the corresponding variables.
Notion of gradient as a 1-form gives us a nice alternate way of writing basis 1-forms: Recall basis defined via

\[ \tilde{\omega}^a(\tilde{e}_p) = \delta^a_p \]

We also have \[ \partial_p x^a = \delta^a_p \]

\[ \equiv \tilde{d}x^a(\tilde{e}_p) \]

Thus, \[ \tilde{\omega}^a = \tilde{d}x^a \] Basis 1-forms are gradient of coordinates.

Picture of a 1-form: Draw level surfaces of a function. The closer the surfaces, the "larger" the 1-form associated with its gradient

\[ h(x, y) = \text{height} \]

\[ \Delta \tilde{x} = \text{displacement vector} \]

\[ \delta h = 1\text{-form of height} \]

\[ \delta h(\Delta \tilde{x}) = \Delta x^a \partial_a h \equiv \Delta h \]

- Proportional to number of contours pierced by arrow.

Nice aspect of this picture is that basis 1-forms \[ \tilde{d}x^a \] are level surfaces of constant coordinate.

Will be very useful when we discuss fluxes!
Metric with both "slots" filled yields a number:

\[ \vec{A} \cdot \vec{B} = \vec{\eta} (\vec{A}, \vec{B}) \]

What about with only one slot filled?

\[ \vec{\eta} (\vec{A}, -) = \text{object that takes 1 vector produces a number} \]

\[ = 1-\text{form} \]

Definition: \( \vec{\Delta}(\cdot) = \vec{\eta} (\vec{A}, -) \)

Components:

\[ A_{\alpha} = \vec{\Delta} (\vec{e}_\alpha) \]
\[ = \vec{\eta} (\vec{A}, \vec{e}_\alpha) \]
\[ = \vec{\eta} (A^p \vec{e}_p, \vec{e}_\alpha) \]

\[ \Rightarrow \quad A_{\alpha} = \eta^{\alpha \rho} A^\rho \]

Metric converts vector into 1-form by "lowering" index.

Invertible procedure: Define \( \eta^{\alpha \beta} \) by \( \delta^\alpha {}_\beta = \eta^{\alpha \rho} \eta_{\rho \beta} \). (Note, \( \eta^{\alpha \beta} \) has the same matrix representation as \( \eta_{\rho \beta} \) !)

Then,

\[ A^\alpha = \eta^{\alpha \rho} A_\rho \]

1-form \( \vec{\Delta} \) is dual to vector \( \vec{A} \).

Notice:

\[ \vec{A} \cdot \vec{B} = \vec{\eta} (\vec{A}, \vec{B}) = \vec{\Delta} (\vec{B}) = \vec{\Delta} (\vec{A}) \]
\[ = \eta^{\alpha \rho} A^\alpha B^\rho = A_{\alpha} B^\alpha = A^\alpha B_\alpha \]
\[ = \eta^{\alpha \rho} A_\alpha B^\rho \]

All the same, all invariant. Distinction among objects getting silly!
Dualism tells us that vectors are themselves tensors. They map 1-forms to Lorentz scalars:

$$\bar{A}(\bar{p}) = A^\mu p_\mu = A^{\alpha \mu \nu} p^{\alpha \nu} = \bar{\bar{A}}(\bar{p}) = \bar{\bar{\bar{p}}}(\bar{A})$$

Operationally, the distinction between "operator" and "operantee" is becoming irrelevant. Index notation highlights fundamental equality of the two species of objects.

Vector: (1) tensor, maps one 1-form into scalars.

(1) tensor, maps M 1-forms into scalars.

Further:

A tensor of type \((M \ N)\) is a linear mapping of \(M\) 1-forms and \(N\) vectors to Lorentz scalars.

Such an object is represented in some frame by an object with \(M\) "upstairs" indices and \(N\) "downstairs" indices.

Silly distinction, though, since metric lets us raise + lower indices.

\[
\begin{align*}
(M \ N) & \rightarrow (M-1 \ N+1) \quad R_{\alpha \beta \gamma \delta} = R^{\mu \nu} R^\mu_{\alpha \beta} R^\nu_{\gamma \delta} \\
\text{raise} & \rightarrow (M+1 \ N-1) \quad S^\alpha_{\rho \delta} = R^\alpha_{\mu \nu} S_{\mu \rho \delta}
\end{align*}
\]
Do we need a basis for tensors?
\[ \bar{\eta} = \eta^{\alpha \beta} \bar{\epsilon}_{\alpha \beta} = \eta^{\alpha \beta} \bar{\epsilon}_{\alpha \beta} \]

Imagine so. Let's examine these objects:

- We know
  \[ \eta^{\alpha \nu} = \bar{\eta} (\bar{e}_\mu, \bar{e}_\nu) \]
  \[ = \eta^{\alpha \beta} \bar{\omega}_{\beta \nu} (\bar{e}_\mu, \bar{e}_\nu) \]

- Requires
  \[ \bar{\omega}^{\alpha \beta} (\bar{e}_\mu, \bar{e}_\nu) = \delta^{\alpha \mu} \delta^{\beta \nu} \]
  \[ = \bar{\omega}^\alpha (\bar{e}_\mu) \bar{\omega}^\beta (\bar{e}_\nu) \]

\( \bar{\omega}^{\alpha \beta} \) is just the "outer product" or "tensore product" of a basis 1-forms!

- \( \bar{\omega}^{\alpha \beta} = \bar{\omega}^\alpha \otimes \bar{\omega}^\beta \)

Likewise,

- \( \bar{e}^{\alpha \beta} = \bar{e}^\alpha \otimes \bar{e}^\beta \)

Generalizes:
\[ \bar{R}^{\alpha \beta \gamma \delta} = R^{\mu \nu \rho \sigma} \bar{e}_\mu \otimes \bar{\omega}^\beta \otimes \bar{\omega}^\delta \otimes \bar{\omega}^\gamma \]

A lot of baggage! Typically stick with components:
It's understood basis tensors are coming along for the ride.

Note: Using basis tensors, transformations are obvious.

\[ T^{\alpha \beta} \bar{\delta}^\gamma \bar{\epsilon} = T^{\alpha \nu} \delta^\nu \otimes \bar{\epsilon}^\gamma \]
\[ \Lambda^\mu \Lambda^\nu \Lambda^\sigma \Lambda^\delta \Lambda^\beta \Lambda^\gamma \Lambda^\delta \Lambda^\epsilon \]
One place where it is useful to remember existence of basis tensors: Derivatives of tensors.

Trajectory parameterized by $\tau$, defines $\bar{u} = \frac{dx}{d\tau}$.

Now, imagine a tensor field fills all of space:

$$\bar{T} = T^{\alpha}_{\beta} \bar{e}_\alpha \otimes \bar{\bar{e}}^\beta$$

How does this tensor vary along the trajectory?

Start with the old-fashioned definition of a derivative:

$$\frac{d\bar{T}}{d\tau} = \lim_{\Delta \tau \to 0} \frac{\bar{T}(\tau + \Delta \tau) - \bar{T}(\tau)}{\Delta \tau}$$

$$= \frac{d\bar{T}^\alpha}{d\tau} \bar{e}_\alpha \otimes \bar{\bar{e}}^\beta$$

basis is constant... now! It won't be later, and that will make things a bit messy.

$$\frac{d\bar{T}^\alpha}{d\tau} = \alpha^\gamma \frac{d\bar{T}^\alpha}{d\tau}$$

But this is also a tensor - the gradient! (components of)

$$\nabla \bar{T} = \gamma^{\gamma \beta} \bar{T}^\alpha_{\beta} \bar{e}_\alpha \otimes \bar{\bar{e}}^\gamma \otimes \bar{\bar{e}}^\beta$$

Then, $\frac{d\bar{T}}{d\tau} = \bar{u} \cdot \nabla \bar{T} \quad \text{HORRIBLE with him!}$

$$= \nabla \bar{u} \bar{T}$$

indicates contracting or gradient index.
Physics again! Quantities we've introduced so far: good for kinematics of particles:
\[ \vec{u} = ( \hat{u}_x, \hat{u}_y, \hat{u}_z ) \]
\[ \vec{u} \cdot \vec{u} = 1 \] (Note: not good for a photon!)
\[ \vec{p} = m \vec{u} \]
\[ \vec{p} = (E, \vec{p}) \]
\[ \vec{p} \cdot \vec{p} = -m^2 \]
\[ = \hbar \omega \left( \hat{a}^\dagger, \hat{a} \right) \] for a photon. \( \vec{u} \) not defined!
\[ \hat{a} = \text{dir of propagation}. \]

How do we describe more interesting matter?

Simplest continuum form of matter: dust.

Define: a collection of non-interacting particles (no pressure!) at rest in some inertial frame.

Start by examining dust in this frame. Simplest characterization: How many particles do we have per unit volume?

\[ n_0 = \text{number density in the rest frame of dust} \] (Density of an "element")

Now, move into a different reference frame. Total number of particles in "element" must be invariant - count scalar, but volume contracts.

\[ n = \text{density in new frame} \]
\[ = \gamma n_0 = \frac{n_0}{\sqrt{1 - v^2}} \]
\[ \gamma = \frac{1}{\sqrt{1 - v^2}} \]
In this IRF, dust is moving; can define flux——number of particles crossing unit area in unit time.

\[ \mathbf{N} = n_x = \gamma n_o \mathbf{x} \]

These quantities are screaming to be combined into a 4-vector!

\[ \tilde{N} = (n, n_x) = n_0 (\delta, \delta_x) = n_0 \tilde{n} \rightarrow \text{ "Number flux 4-vector"} \]

Notice: \( n_0 = \sqrt{-\tilde{N} \cdot \tilde{N}} \)

rest density maps to "magnitude" of \( \tilde{N} \).

Systematic way to pick out flux across a surface:
Recall \( \tilde{d}x^\alpha \) = 1-form describing surfaces at unit ticks of coordinate \( x^\alpha \). Then,

\[ < \tilde{d}x^\alpha, \tilde{N} > = \text{flux in the } x^\alpha \text{ direction} \]

Note: \( < \tilde{d}t, \tilde{N} > = n_0 = n \).

Density is just "flux" in the time direction!

\[ \# \text{area-time} = \# \text{volume when } c = 1. \]

More generally: define some surface as the solution to

\[ \psi(t, x, y, z) = \text{const} \]

\( \tilde{d}\psi = \text{normal 1-form to that surface} \)

\( \tilde{n} = \tilde{d}\psi / \sqrt{|\tilde{d}\psi \cdot \tilde{n}|} = \text{unit normal 1-form} \)

\[ < \tilde{\psi}, \tilde{n} > = \tilde{n}_\alpha N^\alpha = \text{flux in } \tilde{n} \text{ direction.} \]
Conservation: The flux at must come at the expense of density already there: \[
\frac{\partial n}{\partial t} = - \nabla \cdot \mathbf{N}
\]

\[
\rightarrow \oint_{\partial V} N^2 = 0
\]

Integral form of conservation law: Begin by considering some particular inertial reference frame. Intuitively, this implies that

\[
\frac{\partial}{\partial t} \int_{V^3} N^0 \, dV = - \int_{\partial V^3} N \cdot d\mathbf{a}
\]

\[ V^3 \equiv \text{some 3-volume} \]

\[ \partial V^3 \equiv \text{boundary of that 3-volume} \]

In words: the rate of change of the number of particles in a 3-volume equals the integral of the flux through the boundary of that 3-volume.

Can we put this in geometric, covariant language?