MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
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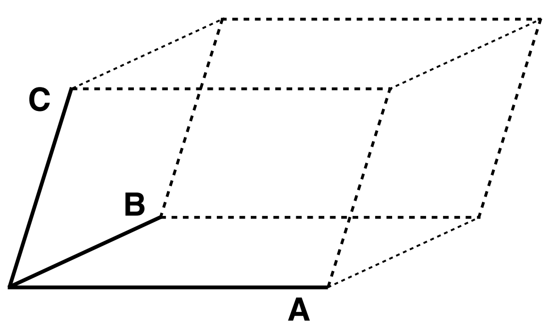
LECTURE 4
SPACETIME AND GEOMETRY PART IV

4.1 Volumes and volume integrals

The previous lecture ended wondering whether we could define a notion of a volume integral that would be useful for us, motivated by the idea that we'd like to be able to "integrate up" a differential expression of a conservation principle to describe the way some quantity (number, charge; perhaps energy and momentum) are conserved in some specified volume. The usual framework is manifestly *not* Lorentz covariant: conservation laws tend to balance the rate at which the amount of some conserved "stuff" in a volume changes against the flow of that "stuff" through the volume's sides. This explicitly depends on defining a time (in order that rates have useful meaning) and space (to define the geometry of the volume and its sides).

Let's set aside concerns about Lorentz covariance for a few moments and just think about how to formulate these integrals using a framework that is as general and coordinate-independent as possible. For intuition, think about a 3-volume: consider the parallelepiped whose sides are defined by the 3-vectors **A**, **B**, and **C**:



A really nice way to compute the volume of this figure is by using the dot product and cross product associated with 3-vectors:

$$\text{3-volume} = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) \ . \tag{4.1}$$

These three forms are totally equivalent, and can be expressed very nicely using a matrix determinant, which in turn can be expressed quite neatly using the Levi-Civita symbol:

$$\text{3-volume} = \begin{vmatrix} A^1 & A^2 & A^3 \\ B^1 & B^2 & B^3 \\ C^1 & C^2 & C^3 \end{vmatrix} = \epsilon_{ijk} A^i B^j C^k \ . \tag{4.2}$$

The Levi-Civita symbol is the totally antisymmetric symbol, defined as

$$\begin{aligned} \epsilon_{123} = \epsilon_{231} = \epsilon_{312} &= +1 \quad \text{(All even permutations of 123)} \\ \epsilon_{132} = \epsilon_{321} = \epsilon_{213} &= -1 \quad \text{(All odd permutations of 123)} \\ \epsilon_{112} &= 0 \quad \text{(Any index repeated)} \end{aligned} \tag{4.3}$$

In Euclidean 3-space, the Levi-Civita symbol can be regarded as the components of a type $\binom{0}{3}$ tensor. When its 3 "slots" are filled by vectors, the output is the volume of the parallelepiped whose sides are defined by those vectors:

$$V^3(\mathbf{A}, \mathbf{B}, \mathbf{C}) = \boldsymbol{\epsilon}(\mathbf{A}, \mathbf{B}, \mathbf{C}) \ . \tag{4.4}$$

Notice that we only fill two of the slots, what we get is a 1-form:

$$\tilde{\Sigma}(\mathbf{B}, \mathbf{C}) \equiv \boldsymbol{\epsilon}(-, \mathbf{B}, \mathbf{C}) \tag{4.5}$$

or

$$\Sigma_i(\mathbf{B}, \mathbf{C}) = \epsilon_{ijk} B^j C^k \ . \tag{4.6}$$

The magnitude of this 1-form is the surface area corresponding to the face spanned by the two vectors this tensor acts upon. Bearing in mind the idea that a 1-form corresponds to the level surface of a function, we can see that the idea that a volume's face is described geometrically by a 1-form makes a lot of sense. (We can always construct the "normal" vector with which you may be more familiar by raising an index with the metric. In Euclidean 3-space with Cartesian coordinates, the metric's components are just the identity matrix, so this is a rather trivial operation.)

With these notions, we can rigorously spell out Gauss's theorem in geometric language: for some 3-vector field $\mathbf{F}$,

$$\int_{V^3} (\boldsymbol{\nabla} \cdot \mathbf{F})\, dV = \oint_{\partial V^3} \mathbf{F}(d\tilde{\Sigma}) \ . \tag{4.7}$$

Here, V^3 is some 3-volume, and ∂V^3 is the boundary of that 3-volume. To construct the volume element dV and the area element 1-form $d\tilde{\Sigma}$, we need to imagine that at each point we integrate over a differential triple $d\mathbf{x}_1$, $d\mathbf{x}_2$, $d\mathbf{x}_3$. Then,

$$\begin{aligned} dV &= \epsilon_{ijk} dx_1^i dx_2^j dx_3^k \ , \\ d\Sigma_i &= \epsilon_{ijk} dx_i^j dx_2^k \ . \end{aligned} \tag{4.8}$$

Note, the particular differential legs that go into $d\Sigma_i$ will depend on the orientation of ∂V^3. For example, a particularly "nice" choice of triple could be $d\mathbf{x}_1 = dx\,\mathbf{e}_x$, $d\mathbf{x}_2 = dy\,\mathbf{e}_y$, $d\mathbf{x}_3 = dz\,\mathbf{e}_z$. With this choice, $dV = dx\,dy\,dz$, and $d\Sigma_i$ will have magnitude $dx\,dy$, $dy\,dz$, $-dx\,dz$, etc, depending upon the detailed geometry of each surface element.

The spacetime generalization of all this follows quite naturally. Consider the *4-volume* enclosed by a parallelepiped in spacetime with sides $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$: it is simply given by

$$\text{4-volume} \equiv V^4 = \epsilon_{\alpha\beta\gamma\delta} A^\alpha B^\beta C^\gamma D^\delta \ . \tag{4.9}$$

We are using the 4-dimensional Levi-Civita symbol, for which $\epsilon_{0123} = +1$, and likewise for all even permutations of 0123; $\epsilon_{1023} = -1$, and likewise for all odd permutations of 0123; and $\epsilon_{0023} = 0$, and likewise whenever an index is repeated.

The surface "area" of each face of the parallelepiped has the dimensions of 3-volume, and can be regarded as a spacetime 1-form. For example, two of the faces have the form

$$\Sigma_\alpha = \epsilon_{\alpha\beta\gamma\delta} B^\beta C^\gamma D^\delta \ ; \tag{4.10}$$

the other six faces involve different combinations of 3 of the 4 defining vectors. Gauss's theorem then carries over to the generalization

$$\int_{V^4} (\partial_\alpha F^\alpha)\, dV^4 = \oint_{\partial V^4} F^\alpha\, d\Sigma_\alpha \ , \tag{4.11}$$

where

$$dV^4 = \epsilon_{\alpha\beta\gamma\delta} dx_0^\alpha \, dx_1^\beta \, dx_2^\gamma \, dx_3^\delta \, ,$$
$$d\Sigma_\alpha = \epsilon_{\alpha\beta\gamma\delta} dx_1^\beta \, dx_2^\gamma \, dx_3^\delta \, . \tag{4.12}$$

Similar to the 3-space example, the particular differential legs that go into $d\Sigma_\alpha$ will depend on the orientation of ∂V^4.

4.2 Integral formulation of conservation laws in spacetime

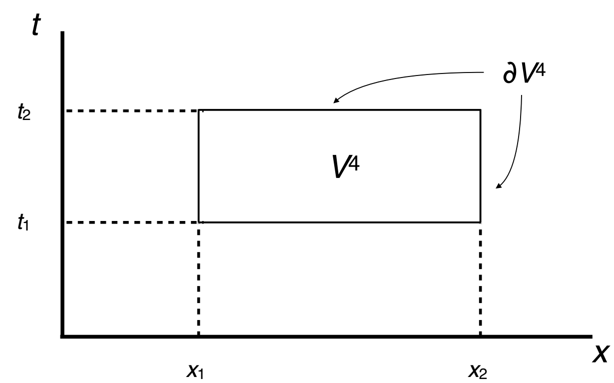
This has all been fairly abstract up to this point. Let's make it concrete by considering the number flux 4-vector for dust:

$$\int_{V^4} (\partial_\alpha N^\alpha) \, dV^4 = \oint_{\partial V^4} N^\alpha \, d\Sigma_\alpha \, . \tag{4.13}$$

The differential conservation law we deduced for this quantities was $\partial_\alpha N^\alpha = 0$, so the covariant integral conservation law is

$$\boxed{\oint_{\partial V^4} N^\alpha d\Sigma_\alpha = 0} \tag{4.14}$$

We emphasize that Eq. (4.14) holds in *all* reference frames: it is a completely covariant integral formulation of the conservation of number in some spacetime volume. However, specific inertial observers will split up the volume V^4 in different ways. Let's look at this equation from the perspective of a specific frame, for which the volume looks as follows:



(This is obviously just a 2-dimensional slice of the 4-volume.) When we evaluate the integral (4.14), there will be 8 contributions, corresponding to the 8 different faces of the 4-cube:

$$0 = \oint_{\partial V^4} N^\alpha d\Sigma_\alpha = \int_{t=t_2} N^0 \, dx \, dy \, dz - \int_{t=t_1} N^0 \, dx \, dy \, dz + \int_{x=x_2} N^1 \, dt \, dy \, dz - \int_{x=x_1} N^1 \, dt \, dy \, dz + \ldots \tag{4.15}$$

Now consider the limit $t_2 \to t_1 + dt$, and rearrange:

$$\int_{t=t_1+dt} N^0 \, dx \, dy \, dz - \int_{t=t_1} N^0 \, dx \, dy \, dz = -dt \left[\int_{x=x_2} N^1 \, dy \, dz - \int_{x=x_1} N^1 \, dy \, dz + \ldots \right] . \tag{4.16}$$

Recalling that, in this frame $N^0 = n$, the number density of dust, and $N^{1,2,3}$ are the components of $\mathbf{n}$, the number flux 3-vector, the above equation becomes

$$\frac{\partial}{\partial t} \int_{V^3} n \, dx \, dy \, dz = - \oint_{\partial V^3} \mathbf{n} \cdot d\mathbf{a} \, . \tag{4.17}$$

This is exactly the rule that we wrote down earlier, demonstrating that Eq. (4.14) is entirely equivalent to this rule, but is written in a *covariant* manner — meaning that it is a formulation that works in all IRFs.

4.3 Electrodynamics: Field equations and equation of motion

A quantity which shares some of the mathematical structure of the number-flux 4-vector is the electric current:

$$\vec{J} \doteq (\rho, \mathbf{J}) \; . \tag{4.18}$$

The continuity equation of electrodynamics takes the form $\partial_\alpha J^\alpha = 0$ in our language. Integrating this over some V^4 and applying Gauss's theorem tells us that

$$\oint_{\partial V^4} J^\alpha d\Sigma_\alpha = 0 \; . \tag{4.19}$$

Examining Eq. (4.19) in some specified IRF, this equation tells us how the charge contained in a volume changes in response to currents that flow into and out of that volume.

Part of what makes the electric current so important is that it acts as the source for electric and magnetic fields. Working in Cartesian coordinates in 3-space (so that whether indices are "upstairs" or "downstairs" is irrelevant — the metric which raises and lowers is just the identity), the field equations in component form are given by

$$\begin{aligned}
\partial_i E_i = 4\pi\rho &= 4\pi J^0 && (4.20)\\
\epsilon_{ijk}\partial_j B_k - \partial_0 E_i &= 4\pi J^i && (4.21)\\
\epsilon_{ijk}\partial_j E_k + \partial_0 B_i &= 0 && (4.22)\\
\partial_i B_i &= 0 \; . && (4.23)
\end{aligned}$$

You presumably learned about this in another course, perhaps without the 4πs but with factors of ϵ_0 and μ_0; the difference is just an uninteresting choice of units. The 6 components of the electromagnetic field are perfectly suited to be packaged into an antisymmetric 2nd-rank tensor; we define components in a particular frame according to

$$F^{0i} = E^i = -F^{i0} \; , \qquad F^{ij} = \epsilon^{ijk} B_k \; . \tag{4.24}$$

This object has the matrix representation

$$F^{\mu\nu} \doteq \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & -B_x & 0 \end{pmatrix} \; . \tag{4.25}$$

It's not hard to show that the field equations are given by

$$\begin{aligned}
\partial_\nu F^{\mu\nu} &= 4\pi J^\mu \; , && (4.26)\\
\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} &= 0 \; . && (4.27)
\end{aligned}$$

Just expand these out and you'll see that Eq. (4.26) is equivalent to Eqs. (4.20) and (4.21), and (4.27) is equivalent to (4.22) and (4.23). An alternate way of writing (4.27) is to define a "dual" electromagnetic field tensor,

$$G_{\alpha\beta} = \epsilon_{\alpha\beta\gamma\delta} F^{\gamma\delta} \; ; \tag{4.28}$$

using this, the source-free part of the Maxwell equations becomes

$$\partial_\alpha G^{\alpha\beta} = 0 \; . \tag{4.29}$$

This is quite pretty, though one should bear in mind that we are soon going to use the symbol $G^{\alpha\beta}$ for a totally different (and extremely important) tensor in our relativistic theory of gravity. It's also worth remarking that when one lowers or raises indices, various minus signs come into play thanks to the negative component of the "time-time" piece of the metric. For example,

$$F_{\mu\nu} \doteq \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} . \tag{4.30}$$

Be careful and double check all quantities when working with these tensors; it's easy to introduce stray minus signs.

One of the beautiful features of the electromagnetic field equation as written in this tensorial index notation is that conservation of the source is automatically built into the formalism. To see this, take the divergence of the sourced field equation (4.26):

$$\begin{aligned} 4\pi\partial_\mu J^\mu &= \partial_\mu\partial_\nu F^{\mu\nu} \\ &= \partial_\nu\partial_\mu F^{\nu\mu} \qquad \text{(Dummy indices can be relabeled, so switch } \mu \text{ and } \nu\text{)} && (4.31) \\ &= -\partial_\nu\partial_\mu F^{\mu\nu} \qquad (F^{\mu\nu} \text{ is antisymmetric}) && (4.32) \\ &= -\partial_\mu\partial_\nu F^{\mu\nu} \qquad (\partial_\mu\partial_\nu \text{ is symmetric}) && (4.33) \\ &= 0 \,. && (4.34) \end{aligned}$$

The final equality comes from comparing the first line with the fourth line: the right-hand side is equal to its negative, which is only true if it is zero. The field equations thus automatically enforce conservation of the current vector J^μ.

The trick that the above calculation introduced is often called "symmetry-antisymmetry": whenever an object that is symmetric on exchange of indices is contracted onto an object that is antisymmetric on index exchange, then the result must be zero. Another place where this result comes up is in the equation of motion a charge feels due to an electromagnetic field. In previous physics cources, you probably learned the Lorentz force law $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. This is not wrong, but it is not written in covariant form: we have selected notions of "time" and "space," so this is written for some observer's IRF. The covariant version of this is

$$\frac{dp^\mu}{d\tau} = qF^{\mu\nu}u_\nu \,, \tag{4.35}$$

or

$$a^\mu = \frac{q}{m}F^{\mu\nu}u_\nu \,. \tag{4.36}$$

A lovely feature of our formalism is that this equation automatically builds in the idea that the 4-acceleration is spacetime orthogonal to the 4-velocity. To see this, contract u_μ into both sides of this equation. On the left, we have $a^\mu u_\mu$, something we hope is zero. On the right, we have something proportional to $F^{\mu\nu}u_\mu u_\nu$. The tensor $F^{\mu\nu}$ is antisymmetric on exchange of indices; the combination $u_\mu u_\nu$ is symmetric. Thanks to "symmetry-antisymmetry," the right-hand side is indeed zero; our force law is one allowed by relativistic kinematics.

4.4 The stress-energy tensor

Let's come back to our dust, and now look at something more interesting than just the number of particles per unit volume. Suppose each particle has a mass m; in "normal" units, we would say that each particle has a rest energy mc^2. In the rest frame (and going back to $c = 1$ units), the rest energy density of the dust is given by

$$\rho \equiv \rho_0 = mn_0 \,. \tag{4.37}$$

How about in another frame? Suppose that in frame $\mathcal{O}$, the dust is moving with $\mathbf{v}$ relative to the rest frame. The number density goes up by a factor of γ due to the length contraction of the fiducial volume. The energy per particle also goes up by γ, since in this frame the particles are not at rest. So,

$$\rho = (\gamma m)(\gamma n_0) = \gamma^2 \rho_0 \ . \tag{4.38}$$

This is neither the transformation law of a scalar, nor of a 4-vector component. In fact, m is a component of $\vec{p}$ (it is the energy, at least in one particular frame), and n_0 is a component of $\vec{N}$ (it is the number density in that same frame), so the density that we originally assembled in fact comes out of the tensor product of two vectors. What we've developed is one component of a quantity known as the *stress-energy tensor* for dust:

$$\mathbf{T} = \vec{N} \otimes \vec{p} = n_0 m \, \vec{u} \otimes \vec{u} = \rho_0 \, \vec{u} \otimes \vec{u} \ , \tag{4.39}$$

or

$$T^{\alpha\beta} = \rho_0 u^\alpha u^\beta \ . \tag{4.40}$$

The physical meaning of the stress-energy tensor is simple: $T^{\alpha\beta}$ describes the flux of 4-momentum component p^α in the x^β direction. This holds *in general*, not just for dust (which are using because it is simple and illustrative). Let's think about what the components mean the four principle groupings of space and time in some IRF:

$$T^{00} = \text{flux of } p^t \text{ in the } t \text{ direction} = \text{energy density for observers in this IRF} \tag{4.41}$$

$$T^{0i} = \text{flux of } p^t \text{ in the } x^i \text{ direction} = \text{energy flux along } x^i \tag{4.42}$$

$$T^{i0} = \text{flux of } p^i \text{ in the } t \text{ direction} = \text{density of momentum } p^i \tag{4.43}$$

$$T^{ij} = \text{flux of } p^i \text{ in the } x^j \text{ direction} \ . \tag{4.44}$$

Much of what we encounter in physics is not dust. To understand $T^{\alpha\beta}$ for whatever situation we are studying, it is fairly common to understand what the components mean in some IRF, and then "promote" terms to a covariant form.

An important example for the study of gravitational physics is the "perfect" fluid. This is a fluid in which there is no energy flow in the rest frame (meaning, for example, that it does not transport heat) and for which it exerts no lateral stresses in that frame (i.e., T^{ij} is diagonal — there is no viscosity to transport p^i in a direction other than x^i). These simplifications mean that this is a rather fictional fluid; it has been derided as a mathematical description of "dry" water. Nonetheless, for certain applications (especially ones where the goal is to understand gravity), this approximation includes all the terms which make a significant contribution to the physics we wish to study. Section 4.5 in the textbook by Schutz does a very nice job laying out what is needed to go beyond this basic picture; for us, the basic picture is sufficient. A perfect fluid is characterized by two numbers, the energy density in the fluid's rest frame ρ and the *pressure* P which describes momentum flux. In a perfect fluid's rest frame, the stress-energy tensor's components are

$$T^{\alpha\beta} \doteq \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \doteq \text{diag}(\rho, P, P, P) \ . \tag{4.45}$$

This tensor can be written in a totally covariant manner by recognizing that the ρ attaches to the fluid 4-velocity of each element (recall that the components above are in the fluid's rest frame), and that the P attaches to the tensor which projects quantities orthogonal to those velocities (which you worked out on a problem set):

$$\mathbf{T} = \rho \, \vec{u} \otimes \vec{u} + P \, (\boldsymbol{\eta} + \vec{u} \otimes \vec{u}) \tag{4.46}$$

or

$$T_{\alpha\beta} = \rho \, u_\alpha u_\beta + P(\eta_{\alpha\beta} + u_\alpha u_\beta) = (\rho + P) u_\alpha u_\beta + P \eta_{\alpha\beta} \ . \tag{4.47}$$

4.5 Conservation of stress-energy

The stress-energy tensor is the tool that we use to describe how energy and momentum are distributed in spacetime; as such, it is the tool that we use to describe how energy and momentum are conserved for matter and fields that are so distributed. Because energy and momentum are connected into a single object (indeed, what is energy to one observer will be a mixture of energy and momentum to another), we do not treat their conservation laws separately, but combine them into the following form:

$$\boxed{\partial_\alpha T^{\alpha\beta} = 0} \tag{4.48}$$

To really see that this corresponds to conservation of energy and conservation of momentum, we need to look at it in some particular observer's reference frame. Let's choose a frame, and begin with the timelike component of Eq. (4.48):

$$\partial_\alpha T^{\alpha 0} = 0\ ; \tag{4.49}$$

or,

$$\frac{\partial T^{00}}{\partial t} = -\frac{\partial T^{i0}}{\partial x^i}\ . \tag{4.50}$$

In words, the rate of change of energy density T^{00} measured by our observer is equal to the (minus) divergence of the energy flux. This is exactly the form of energy conservation we should expect on intuitive grounds! Integrating both sides over a 3-volume and invoking Gauss[1], we come to an integral formulation of energy conservation:

$$\frac{\partial}{\partial t}\int_{V^3} T^{00} dV^3 = -\int_{\partial V^3} T^{0i} d\Sigma_i\ . \tag{4.52}$$

The rate of change of energy in V^3 is balanced by the flux of energy across the boundary ∂V^3. (An equivalent way to do this would be to integrate

The spatial component of our general law gives us momentum conservation in this frame:

$$\partial_\alpha T^{\alpha j} = 0 \quad \longrightarrow \quad \frac{\partial T^{0j}}{\partial t} = -\frac{\partial T^{ij}}{\partial x^i}\ , \tag{4.53}$$

$$\frac{\partial}{\partial t}\int_{V^3} T^{0j} dV^3 = -\int_{\partial V^3} T^{ij} d\Sigma_i\ . \tag{4.54}$$

The rate of change of momentum component j in V^3 is balanced by the flux of momentum component j across the boundary ∂V^3.

It's worth re-emphasizing that Eq. (4.48) is, in "spacetime thinking," the fundamental conservation law. Energy and momentum conservation, expressed by Eqs. (4.52) and (4.54) follow from that only after splitting spacetime into a particular observer's space and time. They are 100% correct, and very useful in a particular frame, but have not been expressed in a fully covariant way.

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[1] Or, we could integrate (4.48) over a 4-volume, apply Gauss to that and deduce the law

$$\int_{\partial V^4} T^{\alpha\beta} d\Sigma_\beta = 0\ . \tag{4.51}$$

Nothing changes if we do that — analyzing the $\alpha = 0$ component reproduces our result for conservation of energy, and $\alpha = i$ reproduces that for conservation of momentum. This is simply another route to the same results.