Massachusetts Institute of Technology
Department of Physics
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Lecture 8
Application of the Lie derivative; tensor densities; party tricks with the determinant of the metric

This lecture is the last one in which the focus is on developing mathematical tools. We will continue to develop new tensors that tell us about important quantities in spacetime, but after this lecture we will have all the calculational machinery we need to do this.

8.1 Lie derivatives and symmetries

The Lie derivative $\mathcal{L}_{\vec{u}}$ tells us about how a tensorial quantity changes as one moves along the curve whose tangent is $\vec{u}$. In particular, we describe a tensor as being *Lie transported* if $\mathcal{L}_{\vec{u}}(\text{tensor}) = 0$.

Imagine that such a curve exists in our manifold, and it is parameterized by a quantity λ that increases as we move along the curve; the tangent is defined as $u^\alpha = dx^\alpha/d\lambda$. We can change our coordinates such that $x^0 = \lambda$ on the curve, and such that $x^1 = x^2 = x^3 = \text{constant}$ on the curve. With such coordinates,

$$u^\alpha = \frac{dx^\alpha}{d\lambda} \doteq \delta^\alpha{}_0 \;, \tag{8.1}$$

and so in these coordinates $\partial_\mu u^\alpha = 0$ everywhere along this curve.

Now imagine that some tensor is Lie transported. The statement $\mathcal{L}_{\vec{u}}(\text{tensor}) = 0$ which defines Lie transport turns into

$$u^\alpha \partial_\alpha(\text{tensor}) + (\text{combinations of tensor indices contracted on derivatives of } u^\alpha) = 0 \;. \tag{8.2}$$

Since, in these coordinates, u^α has no non-zero derivatives, the equation of Lie transport further simplifies to

$$u^\alpha \partial_\alpha(\text{tensor}) = \frac{\partial(\text{tensor})}{\partial x^0} = 0 \;. \tag{8.3}$$

This means that the tensor *does not change* as we slide along the curve: it is independent of the coordinate x^0 that we introduced.

Recall a foundational principle of tensor analysis: if a tensorial equation holds in one representation, then it holds in *all* representations. If we find that our tensor is independent of some coordinate, then we have shown it is Lie transported in a representation akin to what we have sketched above. This means there must exist some vector $\vec{u}$ along which the tensor is Lie transported in *all* representations. Lie transport gives us a covariant, frame-independent way of characterizing a symmetry of a tensor field.

8.2 Killing vectors and Killing's equation

Suppose the tensor in question is the metric of spacetime. Let us imagine that there exists some vector field $\vec{\xi}$ along which the metric is Lie transported: $\mathcal{L}_{\vec{\xi}}\, g_{\alpha\beta} = 0$. From the discussion above, this tells us that there exists a coordinate x^0 such that $\partial g_{\alpha\beta}/\partial x^0 = 0$: the metric is constant with respect to that coordinate. The converse is also true: if the metric is constant with respect to some coordinate, then a vector $\vec{\xi}$ exists such that the metric is Lie transported along $\vec{\xi}$. (Proof: if $\partial(\text{tensor})/\partial y = 0$ for some coordinate y, then we can define the vector field $\vec{\xi}$ as the one which flows in the direction in which y increases, holding all other coordinates field. By the logic of the above construction, this is nothing more than the Lie derivative along $\vec{\xi}$ in coordinates adapted to this flow, and as such holds in all representations.)

Let us expand the Lie derivative for this situation:

$$\mathcal{L}_{\vec{\xi}}\, g_{\alpha\beta} = 0 \tag{8.4}$$

expands to

$$\xi^\gamma \nabla_\gamma g_{\alpha\beta} + g_{\alpha\gamma}\nabla_\beta \xi^\gamma + g_{\gamma\beta}\nabla_\alpha \xi^\gamma = 0\ . \tag{8.5}$$

But remember that the covariant derivative of the metric is zero. The first term on the left-hand side of (8.5) is zero; on the other two terms, we can move the metric inside the derivatives and lower the indices on the components of $\vec{\xi}$:

$$\boxed{\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0} \tag{8.6}$$

This result is called *Killing's equation*, and $\vec{\xi}$ is known as a Killing vector[1]. We will use Killing vectors as tools for characterizing symmetries of spacetime; from your experience with Lagrangian mechanics, it should not surprise you that they also will play a role in developing a notion of quantities that are conserved as things move about in spacetime.

8.3 Tensor densities

Tensor "densities" are a further category of mathematical objects which will prove useful in our future work. These objects are best regarded as things that transform between different representations almost, but not quite, in the way that tensors transform. When we change representation, we find that they "miss" the correct transformation law by a factor that looks like the determinant of the matrix which effects the transformation.

One of the most important tensor densities is the Levi-Civita symbol. We treated this as a tensor when our focus was special relativity in inertial coordinates, but (as we'll see in a moment), this is rare example of a case where we can do this. Recall how this symbol was defined:

$$\begin{aligned} \tilde{\epsilon}_{\alpha\beta\gamma\delta} &= +1 \quad \text{for indices 0123 and even permutations} \\ &= -1 \quad \text{for odd permutations of 0123} \\ &= 0 \quad \text{for any index repeated.} \end{aligned} \tag{8.7}$$

(We write this with a tilde to indicate that the Levi-Civita symbol is *not* a tensor.)

To show that this is not a tensor, we quote a theorem: Given any 4×4 matrix M whose components are $M^\alpha{}_\mu$,

$$\tilde{\epsilon}_{\alpha\beta\gamma\delta} M^\alpha{}_\mu M^\beta{}_\nu M^\gamma{}_\rho M^\delta{}_\sigma = \tilde{\epsilon}_{\mu\nu\rho\sigma} |M|\ , \tag{8.8}$$

where $|M|$ is the determinant of the matrix M. Let's take in particular $M^\alpha{}_{\bar{\alpha}} = \partial x^\alpha / \partial x^{\bar{\alpha}}$. Doing so tells us what the Levi-Civita symbol is in the new representation:

$$\tilde{\epsilon}_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}} = \left|\frac{\partial x^{\bar{\mu}}}{\partial x^\mu}\right| \tilde{\epsilon}_{\alpha\beta\gamma\delta} \left(\frac{\partial x^\alpha}{\partial x^{\bar{\alpha}}}\right) \left(\frac{\partial x^\beta}{\partial x^{\bar{\beta}}}\right) \left(\frac{\partial x^\gamma}{\partial x^{\bar{\gamma}}}\right) \left(\frac{\partial x^\delta}{\partial x^{\bar{\delta}}}\right)\ . \tag{8.9}$$

If the determinant of the transformation matrix were not present, then this would be tensorial[2]. Instead, we call this a "tensor density of weight 1."

[1] Named for a 19th century German mathematician, Wilhelm Killing, who did a lot of foundational work on Lie algebras independent of Lie, as well as work on non-Euclidean geometries. He was apparently a lovely person, despite his (in English) homicidal surname.

[2] It *is* tensorial if the determinant of the transformation matrix happens to be 1, which in fact is the case when the spacetime is the metric of special relativity in inertial coordinates. This is why it was acceptable for us to treat this symbol as a tensor when we first encountered it.

Another important tensor density is related to the metric. The metric is of course a tensor:

$$g_{\bar\alpha\bar\beta} = g_{\alpha\beta}\left(\frac{\partial x^\alpha}{\partial x^{\bar\alpha}}\right)\left(\frac{\partial x^\beta}{\partial x^{\bar\beta}}\right) . \tag{8.10}$$

Take the determinant of both sides of this:

$$\bar g = \left|\frac{\partial x^{\bar\alpha}}{\partial x^\alpha}\right|^{-2} g . \tag{8.11}$$

We use g, with no indices, for the determinant of the metric; $\bar g$ is the determinant of the metric after changing representation. The determinant of the metric is a tensor density of weight -2.

8.4 Converting tensor densities to tensors

The fact that tensor densities do not transform between representations like tensors makes them difficult to work with. Fortunately, if we have a tensor density of weight w, it is easy to convert it to a true tensor: we just multiply by certain powers of the determinant of the metric:

$$(\text{Tensor density of weight } w)|g|^{w/2} = \text{Tensor} . \tag{8.12}$$

Note the absolute value: g might be negative, and $w/2$ might be half integer. There's no harm in introducing an absolute value here; everything derived above remains true if we replace g by $|g|$.

Perhaps the most important example of a tensor density converted to a tensor is the Levi-Civita symbol. We make a tensor which forms covariant volume operators in this way:

$$\epsilon_{\alpha\beta\gamma\delta} \equiv \sqrt{|g|}\,\tilde\epsilon_{\alpha\beta\gamma\delta} . \tag{8.13}$$

You are in fact already familiar with the effect of this, though you likely came at the result by a different route. Consider the 3-volume element in spherical polar coordinates. The line element is given by $ds^2 = g_{ij}dx^i dx^j = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2$. From this, we deduce that $g = g_{rr}g_{\theta\theta}g_{\phi\phi} = r^4\sin^2\theta$. Defining three principle basis vectors $d\mathbf{x}_1 = dr\mathbf{e}_r$, $d\mathbf{x}_2 = d\theta\mathbf{e}_\theta$, $d\mathbf{x}_3 = d\phi\mathbf{e}_\phi$, we find

$$dV^3 = \sqrt{g}\,\tilde\epsilon_{ijk}\,dx_1^i\,dx_2^j\,dx_3^k = r^2\sin^2\theta\,dr\,d\theta\,d\phi . \tag{8.14}$$

Similar exercises allow us to work out 4-volume elements in given spacetimes, or the 3-volume associated with a particular constant value of some time coordinate in such a spacetime.

8.5 Party trick

The last section presents a fairly miscellaneous, but often quite useful, trick that uses the determinant of the metric. In particular, it turns out that a certain combination of Christoffel symbols can be computed very simply and effectively using g.

The Christoffel combination that is simplified is given by

$$\begin{aligned}
\Gamma^\mu{}_{\mu\alpha} &= g^{\mu\beta}\Gamma_{\beta\mu\alpha} \\
&= \frac{1}{2}g^{\mu\beta}\left(\partial_\mu g_{\alpha\beta} + \partial_\alpha g_{\beta\mu} - \partial_\beta g_{\mu\alpha}\right) \\
&= \frac{1}{2}g^{\mu\beta}\partial_\alpha g_{\mu\beta} .
\end{aligned} \tag{8.15}$$

Note that we *are* summing over μ here; this is not like the homework exercise you did on problem set 3 in which repeated indices were not summed. In going from line 1 to line 2, we are using the form of the Christoffel we worked out a few lectures ago; in going from line 2 to line 3 we use the fact that the combination $(\partial_\mu g_{\alpha\beta} - \partial_\beta g_{\alpha\mu})$ is antisymmetric on exchange of μ and β, but the combination is contracted with the symmetric $g^{\mu\beta}$.

What we will now show is that

$$\frac{1}{2} g^{\mu\beta} \partial_\alpha g_{\mu\beta} = \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|}\right) = \partial_\alpha \left(\ln \sqrt{|g|}\right) \ , \tag{8.16}$$

and thus

$$\Gamma^\mu{}_{\mu\alpha} = \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|}\right) = \partial_\alpha \left(\ln \sqrt{|g|}\right) \ . \tag{8.17}$$

To show this, let's examine the following matrix algebra analysis. Consider some matrix M, and imagine the following variation:

$$\begin{aligned} \delta \ln(\det \mathsf{M}) &= \ln\left[\det\left(\mathsf{M} + \delta\mathsf{M}\right)\right] - \ln\left(\det \mathsf{M}\right) \\ &= \ln\left[\frac{\det\left(\mathsf{M} + \delta\mathsf{M}\right)}{\det \mathsf{M}}\right] \\ &= \ln\left[\det\left(\mathsf{I} + \mathsf{M}^{-1} \cdot \delta\mathsf{M}\right)\right] \ . \end{aligned} \tag{8.18}$$

On the last line, we used the fact that $(\det \mathsf{M})^{-1} = \det \mathsf{M}^{-1}$, as well as the fact that $(\det \mathsf{M})(\det \mathsf{N}) = \det(\mathsf{M} \cdot \mathsf{N})$. The I appearing on that line is the identity matrix.

We now introduce a useful identify: if $\boldsymbol{\epsilon}$ is a "small" matrix, then

$$\det\left(\mathsf{I} + \boldsymbol{\epsilon}\right) \simeq 1 + \mathrm{Tr}(\boldsymbol{\epsilon}) \ , \tag{8.19}$$

where $\mathrm{Tr}(\boldsymbol{\epsilon}) = g^{\alpha\beta}\epsilon_{\alpha\beta}$ is the trace of $\boldsymbol{\epsilon}$. Note that this quantity is a scalar.

So, under the assumption that $\delta\mathsf{M}$ is a very small variation,

$$\delta \ln(\det \mathsf{M}) = \ln\left[\det\left(\mathsf{I} + \mathsf{M}^{-1} \cdot \delta\mathsf{M}\right)\right] = \ln\left[1 + \mathrm{Tr}\left(\mathsf{M}^{-1} \cdot \delta\mathsf{M}\right)\right] = \mathrm{Tr}\left(\mathsf{M}^{-1} \cdot \delta\mathsf{M}\right) \ . \tag{8.20}$$

Now, set the matrix M to the metric components $g_{\alpha\beta}$: $\det \mathsf{M}$ becomes $|g|$, M^{-1} becomes $g^{\alpha\beta}$, and Eq. (8.20) becomes

$$\delta \ln |g| = \mathrm{Tr}\left(g^{\mu\beta} \delta g_{\beta\gamma}\right) = g^{\mu\beta} \delta g_{\mu\beta} \ . \tag{8.21}$$

Imagine that this variation arises from a small variation in the coordinates, δx^α; divide by this variation, take the limit, and find

$$\partial_\alpha \ln |g| = g^{\mu\beta} \partial_\alpha g_{\beta\mu} \ . \tag{8.22}$$

Combining this with our earlier result, we have

$$\Gamma^\mu{}_{\mu\alpha} = \frac{1}{2} g^{\mu\beta} \partial_\alpha g_{\mu\beta} = \partial_\alpha \ln \sqrt{|g|} = \frac{1}{\sqrt{|g|}} \partial_\alpha \sqrt{|g|} \ . \tag{8.23}$$

A particularly useful application of this party trick is to the divergence of a vector:

$$\begin{aligned} \nabla_\alpha A^\alpha &= \partial_\alpha A^\alpha + \Gamma^\alpha{}_{\alpha\beta} A^\beta \\ &= \partial_\alpha A^\alpha + \Gamma^\beta{}_{\beta\alpha} A^\alpha \\ &= \partial_\alpha A^\alpha + \frac{A^\alpha}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|}\right) \\ &= \frac{1}{\sqrt{|g|}} \partial_\alpha \left(\sqrt{|g|} A^\alpha\right) \ . \end{aligned} \tag{8.24}$$

What's really nice about this is that it only involves partial derivatives. This form enables a really lovely way to write Gauss's theorem that works very nicely even in arbitrary curved geometries:

$$\int_{V^4} \left(\nabla_\alpha A^\alpha\right) \sqrt{|g|} d^4x = \int_{V^4} \partial_\alpha \left(\sqrt{|g|} A^\alpha\right) d^4x = \oint_{\partial V^4} A^\alpha \sqrt{|g|} d\Sigma_\alpha \ . \tag{8.25}$$

(Here $d^4x = dx^0\, dx^1\, dx^2\, dx^3$ is the product of the differentials of our 4 spacetime coordinates, in whatever coordinate system is being used.)

Equation (8.25) can be quite valuable when dealing with vector-valued quantities that may be subject to some kind of conservation principle (e.g., number flux or charge/current density). Given that energy and momentum are expressed using the stress-energy energy tensor, and are subject to a nice integral conservation law in flat spacetime, one might wonder — can we generalize this to tensors? Unfortunately, there is no analogous result for the divergence of tensors. An attempt to construct such a generalization fails in two ways:

- First, when we expand the divergence of a tensor, the result is just not as amenable to a nice simplification except under very special circumstances:

$$\nabla_\alpha A^{\alpha\beta} = \partial_\alpha A^{\alpha\beta} + \Gamma^\alpha{}_{\alpha\gamma} A^{\gamma\beta} + \Gamma^\beta{}_{\alpha\gamma} A^{\alpha\gamma} \,. \tag{8.26}$$

  The second term on the right-hand side could be simplified using our result involving the determinant of the metric, but the third term cannot be simplified. The only circumstance when this simplifies is if the tensor $A^{\alpha\beta}$ is antisymmetric, in which case that final term is zero by symmetry-antisymmetry. This unfortunately is rarely useful.

- Even if the divergence can be simplified, *we cannot integrate it up* to get something useful. Consider the stress-energy tensor, $T^{\alpha\beta}$. The divergence of this tells us about conservation of energy and momentum. However, the resulting mathematical object, $\nabla_\alpha T^{\alpha\beta}$ is a 4-vector. In a curved geometry, we cannot "add up" 4-vectors from different regions of an integrand: each point in the integrand has its own tangent space, and so vector-valued contributions at one point cannot be simply combined with contributions at another point.

A corollary of this final point is that many global conservation laws simply do not work in curved spacetime. They may work *under certain circumstances*, but one has to be careful to understand what those circumstances are to make sure that the conservation law has been properly translated to the curved spacetime context.