MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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LECTURE 9
MOTION OF AN UNFORCED BODY IN CURVED SPACETIME

9.1 Motion in curved spacetime

How does a body, subject to no external *non-gravitational* forces, move in spacetime? For intuition, go back to special relativity in inertial coordinates. In this situation, a body is at rest in some set of frames, and moves with constant velocity in all other inertial frames. Its motion in these frames is simply a straight line:

$$x^\alpha = x^\alpha_0 + u^\alpha \tau \ , \tag{9.1}$$

where τ is proper time along this trajectory. The Einstein equivalence principle tells us that (9.1) holds in a local Lorentz frame. Our challenge is to take this idea and translate it into a form that holds more generally.

What does moving in "simply a straight line" really mean? A more complete expression of this idea is that the trajectory at moment $\tau + d\tau$ moves in the same direction in which it was going at moment τ. The *tangent* to the trajectory remains constant over this interval, which means that the trajectory *parallel transports its tangent vector.*

As usual, imagine trajectories that are parameterized by some λ which grows along their worldlines. We define $u^\alpha = dx^\alpha/d\lambda$ as the tangent. If the tangent vector is itself parallel transported along the worldline, then it satisfies

$$u^\alpha \nabla_\alpha u^\beta = 0 \quad \longrightarrow \quad \frac{Du^\alpha}{d\lambda} = 0 \ . \tag{9.2}$$

(The second form of this equation introduces the notation $u^\alpha \nabla_\alpha \equiv D/d\lambda$. We will use this operator occasionally to mean "the covariant derivative along the curve whose tangent is $\vec{u}$.") Expanding the covariant derivative yields

$$u^\alpha \partial_\alpha u^\beta + \Gamma^\beta{}_{\alpha\gamma} u^\alpha u^\gamma = 0 \ , \tag{9.3}$$

which is typically written in the form

$$\frac{du^\beta}{d\lambda} + \Gamma^\beta{}_{\alpha\gamma} u^\alpha u^\gamma = 0 \qquad \text{or} \qquad \frac{d^2 x^\beta}{d\lambda} + \Gamma^\beta{}_{\alpha\gamma} \frac{dx^\alpha}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0 \ . \tag{9.4}$$

Equation (9.4) is known as the *geodesic equation*, and the curves $x^\alpha(\lambda)$ which solve it are *geodesics.*

As a brief but occasionally useful aside, a more general form of the geodesic equation is to allow its normalization to change as it is transported:

$$\frac{Du^\alpha}{d\lambda^*} = \kappa(\lambda^*) u^\alpha \ . \tag{9.5}$$

As you will show on a homework exercise, we can always change our parameterization from λ^* to λ in such a way that the "normal" geodesic equation results. Define $v^\alpha = dx^\alpha/d\lambda$. As you will show on an upcoming problem set, if

$$\frac{d\lambda}{d\lambda^*} = \exp\left[\int_0^\lambda \kappa(\lambda^*) d\lambda^*\right] \ , \tag{9.6}$$

then $v^\alpha \nabla_\alpha v^\beta = 0$ The parameter conversion (9.6) puts the equation in the "standard" geodesic form, in which the tangent vector maintains its normalization as it is transported.

A choice of λ which maintains normalization (i.e., for which $Du^\alpha/d\lambda = 0$) is known as an *affine parameterization.* An affine parameterization corresponds to one which has uniformly spaced "tickmarks" in local Lorentz frames along the worldline. At least for timelike trajectories, a very natural choice for this is the proper time τ. As such, it is very common to see Eq. (9.4) written using τ rather than λ. Note that linear transformations leave affine parameterizations affine: if λ is an affine parameterization, then so is

$$\lambda' = a\lambda + b\ , \tag{9.7}$$

with a and b both constants. Such a reparameterization simply shifts the origin (choice of b), and changes the spacing of tickmarks in each local Lorentz frame (choice of a). The change of spacing is like changing units (e.g., from seconds to minutes).

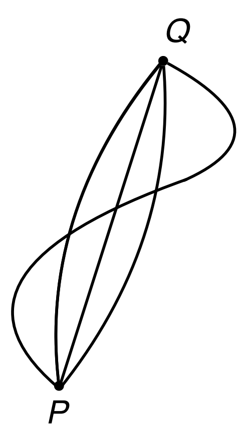
What about motion that *is* subject to an external force? In this case, we simply replace the 0 on the right-hand side of the geodesic equation with an acceleration. Consider a timelike trajectory for which proper time is an appropriate parameter. Our equation of motion in this case becomes

$$\frac{Du^\alpha}{d\tau} = \frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu = a^\alpha\ . \tag{9.8}$$

The acceleration that you put in depends upon the physics of whatever is forcing your body's motion. For instance, if the body has a charge q, mass m, and is in an electromagnetic field with components $F^{\alpha\beta}$, the right-hand side is $(q/m)F^{\alpha\beta}u_\beta$.

9.2 Geodesics via extremization

There is a second route to deriving the geodesic equation, which can be regarded as a rigorous derivation of the idea that "the shortest path between two points is a straight line." Our variant of this to consider the proper time which elapses going from event $\mathcal{P}$ to event $\mathcal{Q}$. Consider every possible path that connects $\mathcal{P}$ to $\mathcal{Q}$, including paths which are physically implausible; a sample of some of these paths might look like



Our only major constraint will be that the paths must be everywhere timelike. Let us compute the proper time $\Delta\tau$ experienced by an observer who moves along one of these paths. We will apply a variational principle in order to find the trajectory for which the accumulated $\Delta\tau$ is *extremal.*

As an aside, you should be able to convince yourself that the extremum you wish to compute is the *maximum* of accumulated proper time, rather than a minimum. If this is not clear, consider a path that consists of multiple segments on which the observer moves at nearly light speed in the frame which sees $\mathcal{P}$ and $\mathcal{Q}$ as they are drawn here. Such a path must include very large accelerations. Supposing that the observer survives these accelerations, the accumulated proper time on each segment will be very small thanks to time dilation. We can find a large number of such (highly accelerated) trajectories with accumulated proper time approaching zero, which is the minimum possible. The minimum of $\Delta\tau$ is thus not unique; the unique trajectory found by variational methods must be the maximum.

Consider each path to have an affine parameter λ that increases from $\mathcal{P}$ to $\mathcal{Q}$; the accumulated proper time for an observer who follows one of these paths is

$$\Delta\tau = \int_{\mathcal{P}}^{\mathcal{Q}} d\lambda \sqrt{-g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}} \ . \tag{9.9}$$

This follows from the fact that $ds^2 = -d\tau^2$ within each local Lorentz frame adapted to this timelike observer.

We now will consider variations in this path, though holding the endpoints (events $\mathcal{P}$ and $\mathcal{Q}$) fixed. To facilitate this analysis, we define

$$f \equiv g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\alpha}{d\lambda} \ , \tag{9.10}$$

and then consider variations in $\Delta\tau$ as we we vary the path:

$$\begin{aligned} \delta(\Delta\tau) &= \int_{\mathcal{P}}^{\mathcal{Q}} d\lambda\, \delta\left[(-f)^{1/2}\right] \\ &= -\frac{1}{2}\int_{\mathcal{P}}^{\mathcal{Q}} \frac{\delta f}{(-f)^{1/2}}\, d\lambda \ . \end{aligned} \tag{9.11}$$

Let us now make a choice for λ: we set $\lambda = \tau$ (proper time along that particular trajectory), so that $dx^\alpha/d\lambda$ is the 4-velocity u^α, and $f = g_{\alpha\beta}u^\alpha u^\beta = -1$. Our variational integral is then

$$\delta(\Delta\tau) = -\frac{1}{2}\int \delta f\, d\tau \ . \tag{9.12}$$

Our goal will be to find the trajectory for which this variation is zero; that will identify the trajectory which gives us an extremum in $\Delta\tau$. To facilitate this, let's make a further definition: we define

$$I = \frac{1}{2}\int_{\mathcal{P}}^{\mathcal{Q}} \left(g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right) d\tau \tag{9.13}$$

as an action describing a body's motion from $\mathcal{P}$ to $\mathcal{Q}$; then, $\delta(\Delta\tau) = \delta I$.

We now examine how this action varies as we vary the trajectory, and look for the trajectory for which this variation is *stationary* (i.e, for which $\delta I = 0$ when $\delta x^\alpha \neq 0$). We take $x^\alpha \to x^\alpha + \delta x^\alpha$, $g_{\alpha\beta} \to g_{\alpha\beta} + \delta x^\gamma \partial_\gamma g_{\alpha\beta}$, and impose the boundary condition that $\delta x^\alpha = 0$ at the endpoints of each trajectory. Doing so, we find

$$\delta I = \frac{1}{2}\int_{\mathcal{P}}^{\mathcal{Q}} d\tau \left[\partial_\gamma g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\delta x^\gamma + g_{\alpha\beta}\frac{d(\delta x^\alpha)}{d\tau}\frac{dx^\beta}{d\tau} + g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d(\delta x^\beta)}{d\tau}\right] \ . \tag{9.14}$$

Let us integrate the second term on the right-hand side by parts:

$$\begin{aligned} \frac{1}{2}\int_{\mathcal{P}}^{\mathcal{Q}} d\tau \left[g_{\alpha\beta}\frac{d(\delta x^\alpha)}{d\tau}\frac{dx^\beta}{d\tau}\right] &= -\frac{1}{2}\int_{\mathcal{P}}^{\mathcal{Q}} d\tau \left[g_{\alpha\beta}\frac{d^2x^\beta}{d\tau^2} + \frac{dg_{\alpha\beta}}{d\tau}\frac{dx^\beta}{d\tau}\right]\delta x^\alpha \\ &= -\frac{1}{2}\int_{\mathcal{P}}^{\mathcal{Q}} d\tau \left[g_{\alpha\beta}\frac{d^2x^\beta}{d\tau^2} + \partial_\gamma g_{\alpha\beta}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right]\delta x^\alpha \ . \end{aligned} \tag{9.15}$$

To get the first line, we discarded a boundary term, using the fact that $\delta x^\alpha = 0$ at the limits $\mathcal{P}$ and $\mathcal{Q}$. For the second line, we used the fact that $d/d\tau = (dx^\gamma/d\tau)\partial_\gamma$. The third term on the right-hand side can be manipulated similarly:

$$\frac{1}{2}\int_{\mathcal{P}}^{\mathcal{Q}} d\tau \left[g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d(\delta x^\beta)}{d\tau}\right] = -\frac{1}{2}\int_{\mathcal{P}}^{\mathcal{Q}} d\tau \left[g_{\alpha\beta}\frac{d^2x^\alpha}{d\tau^2} + \partial_\gamma g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\gamma}{d\tau}\right]\delta x^\beta \,. \tag{9.16}$$

Let us now put all the pieces together, and write $dx^\alpha/d\tau = u^\alpha$:

$$\begin{aligned}\delta I = \frac{1}{2}\int_{\mathcal{P}}^{\mathcal{Q}} d\tau \Big[&\partial_\gamma g_{\alpha\beta} u^\alpha u^\beta \delta x^\gamma \\ &-\partial_\gamma g_{\alpha\beta} u^\beta u^\gamma \delta x^\alpha - \partial_\gamma g_{\alpha\beta} u^\gamma u^\alpha \delta x^\beta \\ &-g_{\alpha\beta}\frac{du^\beta}{d\tau}\delta x^\alpha - g_{\alpha\beta}\frac{du^\alpha}{d\tau}\delta x^\beta\Big] \,.\end{aligned} \tag{9.17}$$

We will cycle the dummy indices in this formula in the following way in order to facilitate grouping the terms in a particularly convenient and useful way:

- First line: Leave the indices as is.
- Second line, left term: $\alpha \to \gamma$, $\beta \to \alpha$, $\gamma \to \beta$.
- Second line, right term: $\alpha \to \beta$, $\beta \to \gamma$, $\gamma \to \alpha$.
- Third line, left term: $\alpha \to \gamma$, $\beta \to \alpha$.
- Third line, right term: $\beta \to \gamma$.

The result of this exercise allows us to write

$$\delta I = -\int_{\mathcal{P}}^{\mathcal{Q}} d\tau \left[g_{\gamma\alpha}\frac{du^\alpha}{d\tau} + \frac{1}{2}\left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}\right)u^\alpha u^\beta\right]\delta x^\gamma \,. \tag{9.18}$$

We recognize the second term in the square brackets here as $\Gamma_{\gamma\alpha\beta}$, so

$$\delta I = -\int_{\mathcal{P}}^{\mathcal{Q}} d\tau \left[g_{\gamma\alpha}\frac{du^\alpha}{d\tau} + \Gamma_{\gamma\alpha\beta}u^\alpha u^\beta\right]\delta x^\gamma \,. \tag{9.19}$$

We require $\delta I = 0$ for any δx^γ. This is the case if the term in square brackets is zero; hitting this with $g^{\mu\gamma}$, we see that extremization leads us to

$$\frac{du^\mu}{d\tau} + \Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta = 0 \,. \tag{9.20}$$

This is the geodesic equation yet again.

9.3 Recap and yet another form

Geodesics generalize the notion of "straight" trajectories which describe inertial motion in special relativity to curves spacetime. The technique of parallel transport expresses the idea that since no force acts, the tangent to the trajectory moves parallel to itself (i.e., it is parallel transported). In special relativity, an accelerated trajectory always accumulates less proper time between two events than the inertial trajectory that connects these events. Extremization of accumulated proper time shows that geodesic motion expresses this concept in a curved spacetime. Geodesics thus generalize the notion of inertial motion for bodies moving in a curved spacetime.

It is useful to re-organize the geodesic equation once more by rewriting it in terms of 4-momentum, rather than 4-velocity:

$$u^\alpha \nabla_\alpha u^\beta = 0 \quad \to \quad p^\alpha \nabla_\alpha p^\beta = 0 \;, \tag{9.21}$$

which rewrites the geodesic equation in the form

$$m\frac{dp^\beta}{d\tau} + \Gamma^\beta{}_{\alpha\gamma} p^\alpha p^\gamma = 0 \;. \tag{9.22}$$

This becomes particularly useful when we define an interval of affine parameter using the rule $\Delta\lambda = \Delta\tau/m$. Using this, we then have

$$p^\alpha \equiv \frac{dx^\alpha}{d\lambda} \tag{9.23}$$

and

$$\frac{dp^\beta}{d\lambda} + \Gamma^\beta{}_{\alpha\gamma} p^\alpha p^\gamma = 0 \;. \tag{9.24}$$

The reason that this is useful is that we can take the limit $m \to 0$. Massless particles move on light-like trajectories in relativistic kinematics; the very notion of proper time does not make sense along such trajectories. However, if we imagine a limit of a massive body approaching the light-like trajectory, we can consider $m \to 0$ while $\Delta\tau \to 0$, but doing so in such a way that $\Delta\tau/m$ is always constant. Equation (9.24) is a form of the geodesic equation that is perfectly suited for studying light-like trajectories, provided we use (9.23) to relate the 4-momentum associated with the trajectory to its path in spacetime.

9.4 Things that are conserved along geodesics

Along with being a useful form for studying light-like trajectories, Eq. (9.24) is also useful for identifying quantities that are conserved along geodesics. Equation (9.24) can be rewritten $p^\alpha(\nabla_\alpha p^\beta) = 0$; because the metric commutes with the covariant derivative, we can further rewrite this

$$p^\alpha \left(\nabla_\alpha p_\beta\right) = 0 \;, \tag{9.25}$$

which expands to become

$$m\frac{dp_\beta}{d\tau} - \Gamma^\gamma{}_{\beta\alpha} p^\alpha p_\gamma = 0 \;, \tag{9.26}$$

or

$$m\frac{dp_\beta}{d\tau} = \Gamma_{\gamma\beta\alpha} p^\alpha p^\gamma = \frac{1}{2}\left(\partial_\beta g_{\alpha\gamma} + \partial_\alpha g_{\gamma\beta} - \partial_\gamma g_{\beta\alpha}\right) p^\alpha p^\gamma \;. \tag{9.27}$$

Look carefully at the last two terms in (9.27): the derivatives of the metric are antisymmetric on exchange of α and γ, but they are contracted with the symmetric combination $p^\alpha p^\gamma$. These two terms thus do not contribute to the right-hand side of (9.27), and we obtain the following very interesting form of the geodesic equation:

$$m\frac{dp_\beta}{d\tau} = \frac{1}{2}\partial_\beta g_{\alpha\gamma} p^\alpha p^\gamma \;. \tag{9.28}$$

"What is so interesting about this?" you may reasonably ask. Notice that, if the metric is independent of a particular x^β (so that $\partial_\beta g_{\alpha\gamma} = 0$ for some value of β), then p_β is *constant everywhere on the geodesic*:

$$\partial_\beta g_{\alpha\gamma} = 0 \quad \longrightarrow \quad \frac{dp_\beta}{d\tau} = 0 \quad \longrightarrow \quad p_\beta = \text{constant on geodesic trajectories.} \tag{9.29}$$

This should remind you of a result from Lagrangian mechanics: if a Lagrangian does not depend on a coordinate, then the momentum conjugate to that coordinate is constant.

A lecture back, we discussed Killing vectors, and one of our conclusions was that if the metric was independent of a particular coordinate (i.e., it did not vary along that coordinate), then there exists a Killing vector associated with this symmetry. Let us imagine that our spacetime admits a Killing vector $\vec{\xi}$. We can construct a scalar $p^\beta \xi_\beta$; let us see how this scalar evolves along a geodesic:

$$\begin{aligned} \frac{D(p^\beta \xi_\beta)}{d\lambda} &= p^\alpha \nabla_\alpha \left(p^\beta \xi_\beta\right) \\ &= \xi_\beta \left(p^\alpha \nabla_\alpha p^\beta\right) + p^\alpha p^\beta \nabla_\alpha \xi_\beta \, . \end{aligned} \tag{9.30}$$

The first term on the final line of Eq. (9.30) vanishes because p^α solves the geodesic equation — the term in parenthesis equals zero. In the second term, because $p^\alpha p^\beta$ is symmetric under exchange of α and β, we can rewrite $\nabla_\alpha \xi_\beta$ in a way that makes this symmetry manifestly clear:

$$\begin{aligned} \frac{D(p^\beta \xi_\beta)}{d\lambda} &= \frac{1}{2} p^\alpha p^\beta \left(\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha\right) \\ &= 0 \, . \end{aligned} \tag{9.31}$$

The final reduction is because $\vec{\xi}$ is a Killing vector, and thus satisfies Killing's equation, $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$. We thus see that $p^\beta \xi_\beta$ is constant along any geodesic trajectory.

Two very important examples of this are as follows:

- Suppose that, in some choice of coordinates, $\partial_t g_{\alpha\beta} = 0$. We know that there exists a "timelike" Killing vector, $\vec{\xi}^T$, in this spacetime. We also know that p_0 is conserved along all trajectories; to tie this to special relativity intuition, we define the *conserved energy* along the geodesic as $E \equiv -p_0$. (To understand the minus sign, think about the special relativity limit: the "downstairs" component in that limit is $-E$.) We can write this as $E = -p^\beta \xi^T_\beta$, which is a coordinate invariant expression; the form $E = -p_0$ only holds in the coordinates for which $\partial_t g_{\alpha\beta} = 0$.

  - *Caution*: the conserved energy $E = -p_0$ is **not necessarily** the energy that is measured by an observer who measures properties of the geodesic whose 4-momentum is p^α! The measured energy can be found by invoking the Einstein equivalence principle: if we go into a local Lorentz frame around an observer whose 4-velocity is $\vec{u}$, they measure the energy along a geodesic whose 4-momentum is $\vec{p}$ to be $E_{\vec{u}} = -p_\alpha u^\alpha$. We will see a very specific example of the difference between a conserved energy along a geodesic and the energy someone measures associated with this geodesic in a few weeks. It is important to always think carefully about the specific, quantitative meaning of words like "energy" in the context of an analysis.

- Suppose that, in some choice of coordinates, $\partial_\phi g_{\alpha\beta} = 0$. This means that there is an "axial" Killing vector, $\vec{\xi}^\Phi$, in this spacetime. We know that p_ϕ is conserved along all trajectories; we call this the "axial angular momentum," and define $L_z \equiv p_\phi$. We can write this in coordinate invariant form $L_z = p^\beta \xi^\Phi_\beta$.

9.5 A geodesic in a specific spacetime

To wrap up this discussion, let's look at geodesics in a particularly interesting spacetime:

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \, . \tag{9.32}$$

(We will develop this spacetime quite soon as one of our first solutions once we've made the Einstein field equation.) The function Φ which appears here is required to be small ($\Phi \ll 1$). We imagine that it varies in space but not in time: $\Phi = \Phi(x, y, z)$.

Let us consider a slowly moving body in this spacetime: consider a body whose 4-velocity in these coordinates has components $u^\alpha \doteq (dt/d\tau, dx^i/d\tau)$, with $dt/d\tau \gg dx^i/d\tau$ for all i. (This is equivalent to saying that the spatial velocity components $v^i = dx^i/dt$ in the frame in which we have written down this spacetime are all much smaller than the speed of light.) The geodesic equation in this spacetime has the form

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0\ . \tag{9.33}$$

Because we are considering the slow motion limit, the second term in the geodesic equation is dominated by the terms with $\mu = \nu = 0$; the contributions related to the spatial 4-velocity components are much less important than the contributions from $dt/d\tau$. Our geodesic equation, to a high level of accuracy, can be written

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{00}\left(\frac{dt}{d\tau}\right)^2 = 0\ , \tag{9.34}$$

or

$$\frac{d^2x^\alpha}{dt^2} + \Gamma^\alpha{}_{00} = 0\ . \tag{9.35}$$

Let's focus on the spatial components:

$$\frac{d^2x^i}{dt^2} + \Gamma^i{}_{00} = 0\ . \tag{9.36}$$

The Christoffel symbol we need is given by

$$\begin{aligned}\Gamma^i{}_{00} &= \frac{1}{2}g^{i\alpha}\Gamma_{\alpha00}\\ &= \frac{1}{2}g^{i\alpha}\left(\partial_t g_{0\alpha} + \partial_t g_{\alpha0} - \partial_\alpha g_{00}\right)\\ &= -\frac{1}{2}g^{i\alpha}\partial_\alpha g_{00}\ .\end{aligned} \tag{9.37}$$

We have used the fact that the metric does not depend on t to eliminate the ∂_t terms[1]. Using $g^{i\alpha} = (1-2\Phi)^{-1}\delta^{i\alpha}$ plus the fact that $\Phi \ll 1$, we find

$$\begin{aligned}\Gamma^i{}_{00} &= -\frac{1}{2}(1-2\Phi)^{-1}\delta^{ij}\partial_j(1-2\Phi)\\ &= \delta^{ij}\partial_j\Phi + \mathcal{O}(\Phi^2)\ .\end{aligned} \tag{9.38}$$

Our equation of motion in this spacetime is thus

$$\frac{d^2x^i}{dt^2} = -\delta^{ij}\partial_j\Phi\ . \tag{9.39}$$

Geodesic motion in this spacetime says that the body moves through these coordinates with an acceleration given by the negative gradient of the function Φ that appears in the line element (9.32). This is *exactly* the law of motion that describes a body moving under the influence of a Newtonian gravitational potential Φ. We will in fact see quite soon that Eq. (9.32) is a key tool in making sure that the relativistic theory of gravity we develop includes and respects the Newtonian limit.

[1] Using this, we find the timelike component of the geodesic equation yields $d^2t/d\tau^2 = 0$. In other words, in this limit, coordinate time and proper time are the same up to a choice of origin and a choice of units. This is appropriate for a Newtonian limit.