MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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LECTURE 10
SPACETIME CURVATURE

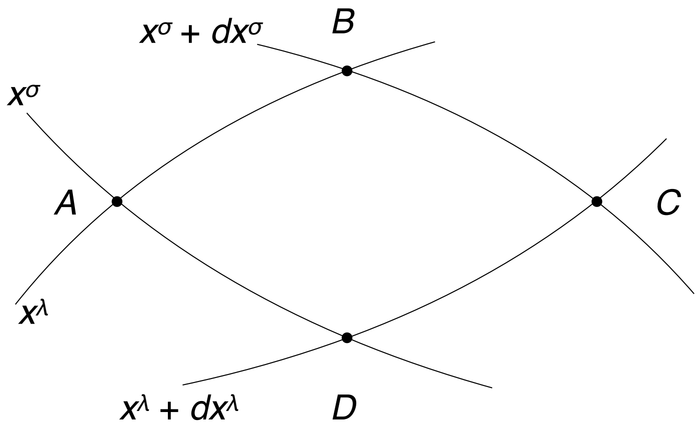
10.1 Quantifying curvature

We have spent much of the past several lectures figuring out how describe tensors and mathematical operations using tensors in curved spacetime manifolds; we have described the motion of freely falling bodies moving through a curved spacetime. However, we have not yet precisely defined what curvature actually *is*. Today we rectify this.

Our concept of curvature centers on the idea that initially parallel trajectories do not remain parallel. This is an important concept to bear in mind (and in Lecture 11, we add some quantitative flesh to this conceptual skeleton), but for cleanly developing a notion of curvature there is an even better tool: take some vector and parallel propagate it around a closed, but finite-sized loop. If the manifold in which this vector moves around has curvature, then the resulting vector will point in a new direction after returning to its starting point (cf. the left-hand panel of the figure below, sketching a vector parallel propagated around a triangle, $A \to B \to C \to A$, that is embedded on the surface of a sphere). If the manifold is *flat*, then the vector will be unchanged when it returns to its starting point (cf. the right-hand panel of the figure below, sketching a vector parallel propagated around a triangle, $A \to B \to C \to A$, that is drawn on a flat surface).

(Such a parallel transport around a closed path is known as a holonomy; those of you interested in the more mathematical side of things can find lots of discussion and details of this operation.) When we are done, the vector has returned to its starting point, and so comparing the initial and final states can be done cleanly using just the components. The basis objects "live" in the same tangent space, and we don't have to worry about how they change during the propagation.

For our analysis, consider parallel transport of a vector with components V^α around the following "diamond," from $A \to B \to C \to D \to A$:



We need only consider two spacetime directions in this analysis. Label one with the index λ, the other with σ. (In the 4 dimensions that we consider for spacetime, there are of course two other; call them x^β and x^γ. We will hold them fixed throughout this analysis.) As we go along the leg from A to B, the coordinate x^λ is held constant, while x^σ increases to $x^\sigma + dx^\sigma$. As we go from B to C, x^λ increases to $x^\lambda + dx^\lambda$; the other is held fixed at $x^\sigma + dx^\sigma$. As we go from C to D, we decrease from $x^\sigma + dx^\sigma$ to x^σ; the other direction is held fixed at $x^\lambda + dx^\lambda$. Finally, returning along D to A, we decrease from $x^\lambda + dx^\lambda$ to x^λ, holding the other coordinate at x^σ.

Let's examine how V^α changes as we parallel transport around these paths. First go from A to B, which we will call "Leg I." We parallel transport with fixed x^λ, in the direction of increasing x^σ. In other words,

$$\nabla_{\vec{e}_\sigma} V^\alpha = 0 \longrightarrow \frac{\partial V^\alpha}{\partial x^\sigma} = -\Gamma^\alpha{}_{\sigma\mu} V^\mu \; . \tag{10.1}$$

Treating this as a differential equation for V^α, we find

$$V^\alpha(B) = V^\alpha_{\rm init} - \int_{\rm I} \Gamma^\alpha{}_{\sigma\mu} V^\mu \, dx^\sigma \; . \tag{10.2}$$

(The notation on the integral means "integrated along Leg I.")

Repeat this to go from B to C ("Leg II"):

$$V^\alpha(C) = V^\alpha(B) - \int_{\rm II} \Gamma^\alpha{}_{\lambda\mu} V^\mu \, dx^\lambda \; . \tag{10.3}$$

From C to D ("Leg III"):

$$V^\alpha(D) = V^\alpha(C) + \int_{\rm III} \Gamma^\alpha{}_{\sigma\mu} V^\mu \, dx^\sigma \; . \tag{10.4}$$

Note the sign flip, which is due to the fact that we are now integrating from $x^\sigma + dx^\sigma$ to x^σ.

Finally, from D back to A ("Leg IV"):

$$V^\alpha_{\rm final} = V^\alpha(C) + \int_{\rm IV} \Gamma^\alpha{}_{\lambda\mu} V^\mu \, dx^\lambda \; . \tag{10.5}$$

We are interested in seeing how the vector changes after going around this loop:

$$\delta V^\alpha \equiv V^\alpha_{\rm final} - V^\alpha_{\rm init} \; . \tag{10.6}$$

As emphasized above, by beginning and ending at the same event, it is easy for us to make this comparison — the initial and final vectors "live" in the same tangent space, so comparing components is sufficient to understand how the vector changes under this operation. Using what we derived above and organizing terms a bit, we have

$$\begin{aligned} \delta V^\alpha &= \int_{\rm IV} \Gamma^\alpha{}_{\lambda\mu} V^\mu \, dx^\lambda - \int_{\rm II} \Gamma^\alpha{}_{\lambda\mu} V^\mu \, dx^\lambda \\ &\quad + \int_{\rm III} \Gamma^\alpha{}_{\sigma\mu} V^\mu \, dx^\sigma - \int_{\rm I} \Gamma^\alpha{}_{\sigma\mu} V^\mu \, dx^\sigma \; . \end{aligned} \tag{10.7}$$

As we go through these contributions, keep in mind that Leg I is evaluated along constant x^λ; Leg III is evaluated along constant $x^\lambda + dx^\lambda$; Leg II is evaluated along constant $x^\sigma + dx^\sigma$; and Leg IV is evaluated along constant x^σ. With this in mind, it follows that

$$\int_{\rm III} \mathcal{I} \, dx^\sigma - \int_{\rm I} \mathcal{I} \, dx^\sigma = \delta x^\lambda \int_{\rm I} \frac{\partial \mathcal{I}}{\partial x^\lambda} dx^\sigma \; , \tag{10.8}$$

$$\int_{\rm IV} \mathcal{I} \, dx^\lambda - \int_{\rm II} \mathcal{I} \, dx^\lambda = -\delta x^\sigma \int_{\rm II} \frac{\partial \mathcal{I}}{\partial x^\sigma} dx^\lambda \tag{10.9}$$

for any integrand $\mathcal{I}$. (We are imaging that δx^λ and δx^σ are sufficiently small that terms of order $(\delta x)^2$ can be discarded; if you want more rigor, please replace $=$ with $\simeq$ and imagine taking a limit at the end of the calculation.) Using this, we can rewrite Eq. (10.7) as

$$\delta V^\alpha = \int_{x^\sigma}^{x^\sigma + dx^\sigma} \delta x^\lambda \frac{\partial}{\partial x^\lambda} \left(\Gamma^\alpha{}_{\sigma\mu} V^\mu \right) dx^\sigma - \int_{x^\lambda}^{x^\lambda + dx^\lambda} \delta x^\sigma \frac{\partial}{\partial x^\sigma} \left(\Gamma^\alpha{}_{\lambda\mu} V^\mu \right) dx^\lambda \; . \tag{10.10}$$

Expanding the derivatives, this becomes

$$\delta V^\alpha = \delta x^\lambda \delta x^\sigma \left[\partial_\lambda \Gamma^\alpha{}_{\sigma\mu} V^\mu - \partial_\sigma \Gamma^\alpha{}_{\lambda\mu} V^\mu + \Gamma^\alpha{}_{\sigma\mu} \partial_\lambda V^\mu - \Gamma^\alpha{}_{\lambda\mu} \partial_\sigma V^\mu \right] \; . \tag{10.11}$$

To reduce those terms involving the partial derivatives of the vector components, remember that they are parallel transported, and so we can write

$$\partial_\lambda V^\mu = -\Gamma^\mu{}_{\lambda\nu} V^\nu \; , \qquad \partial_\sigma V^\mu = -\Gamma^\mu{}_{\sigma\nu} V^\nu \; . \tag{10.12}$$

Inserting this, we find

$$\delta V^\alpha = \delta x^\lambda \delta x^\sigma \left[\left(\partial_\lambda \Gamma^\alpha{}_{\sigma\mu} - \partial_\sigma \Gamma^\alpha{}_{\lambda\mu} \right) V^\mu + \left(\Gamma^\alpha{}_{\lambda\mu} \Gamma^\mu{}_{\sigma\nu} - \Gamma^\alpha{}_{\sigma\mu} \Gamma^\mu{}_{\lambda\nu} \right) V^\nu \right] \; . \tag{10.13}$$

Relabel dummy indices on the last two terms to exchanging μ and ν; the final expression then becomes

$$\delta V^\alpha = R^\alpha{}_{\mu\lambda\sigma} \delta x^\lambda \delta x^\sigma V^\mu \; , \tag{10.14}$$

where we have introduced

$$\boxed{R^\alpha{}_{\mu\lambda\sigma} = \partial_\lambda \Gamma^\alpha{}_{\sigma\mu} - \partial_\sigma \Gamma^\alpha{}_{\lambda\mu} + \Gamma^\alpha{}_{\lambda\nu} \Gamma^\nu{}_{\sigma\mu} - \Gamma^\alpha{}_{\sigma\nu} \Gamma^\nu{}_{\lambda\mu}} \tag{10.15}$$

a quantity known as the *Riemann curvature tensor*. Despite being constructed out of Christoffel symbols, it is truly a tensor: when we change representations, the terms which cause the Christoffels to not transform in a tensorial manner all cancel out.

Notice that the first two contributions to this tensor have the heuristic form "two derivatives of the metric" (bearing in mind that the Christoffel symbol is one derivative of the metric). It is nicely consistent that a quantity which looks like the second derivative of the metric, which we earlier showed cannot be eliminated by going to a local Lorentz frame, is in fact a tensor. The final two terms are of the form "nonlinear first derivative of the metric," a sign that any differential equations we make out of this are going to end up being nonlinear ones.

One can show the calculation we did is equivalent to looking at the commutator of partial derivatives[1]:

$$[\nabla_\lambda, \nabla_\sigma] V^\alpha = R^\alpha{}_{\mu\lambda\sigma} V^\mu \,, \tag{10.16}$$

$$[\nabla_\lambda, \nabla_\sigma] p_\alpha = -R^\mu{}_{\alpha\lambda\sigma} p_\mu \,. \tag{10.17}$$

10.2 Curvature coupling

The way in which we derived the Riemann curvature tensor gives us a flavor of the circumstances in which it matters: curvature tends to enter physics when we look at processes that occur over some kind of extended region. In our case, we had to go around a loop that enclosed a finite region (if we had gone $A \to B \to A$, there would have been no effect), and we ended up with an effect that is proportional to the product of the length of each side of our "parallel transport diamond." This is consistent with the idea that we can make spacetime look like special relativity up to $\mathcal{O}(\delta x)$ from some event, but we cannot do so at $\mathcal{O}(\delta x^2)$.

In our previous lecture, we found that geodesic motion generalizes the notion of free-fall for a body moving in a curved spacetime. The geodesic equation,

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu} u^\mu u^\nu = 0 \,, \tag{10.18}$$

has no curvature in it. We can now see that there's a more-or-less hidden assumption in our discussion of the idea that freely-falling bodies follow geodesic trajectories: we assume that these bodies are so small that the whole body "fits" into a local Lorentz frame. Strictly speaking, when we describe a body as moving along a geodesic, we idealize that body as a zero-size test particle. In reality, we expect that a body of finite size will "couple" to spacetime curvature. For example, the Earth is large enough that it couples to the Riemann curvature tensor associated with the spacetime metric produced by the Sun. This coupling corrects the leading description of the Earth's orbit as following a geodesic, and can be shown to have a small but non-negligible effect, especially when integrated up over many orbits. This in fact is how one describes the precession of the equinoxes in general relativity.

10.3 Symmetries and counting of independent components

The Riemann curvature tensor is a 4-index object, with each index taking n possible values in n dimensional space ($n = 4$ for spacetime, but it will be useful to leave this more general in the following discussion). This suggests naively that it has n^4 different components (256 in spacetime). However, we have not yet looked to see how many symmetries it has.

One symmetry can be seen quite simply: go back to our parallel propagation of the vector around our "curvature diamond," but do so in the opposite direction. Doing so exchanges the roles of λ and σ, and yields a change δV^α with opposite sign. This tells us that

$$R^\alpha{}_{\mu\lambda\sigma} = -R^\alpha{}_{\mu\sigma\lambda} \,. \tag{10.19}$$

Finding that the Riemann tensor is antisymmetric on the last two indices reduces the number of independent components from n^4 to $n^2 \cdot (n(n-1))/2$, which means from 256 to 96 in 4-dimensional spacetime. This is already a significant reduction, but we are just getting started.

[1] **Caution:** the commutator acting on a 1-form is written incorrectly in the 1st edition of Schutz, just in case you happen to have a copy of that text. This error has been corrected in later editions.

To go further, it is useful to lower the first index

$$R_{\alpha\mu\lambda\sigma} = g_{\alpha\nu} R^{\nu}{}_{\mu\lambda\sigma} \ , \tag{10.20}$$

and then expand all the terms in a local Lorentz frame. Within the LLF, all 1st derivatives of the metric vanish; however, 2nd derivatives of the metric do *not* vanish. This amounts to going into a representation in which we keep the $\partial\Gamma$ terms in Eq. (10.15), but we drop the Γ^2 terms. It should be emphasized that we do not *need* to do this. Going to the LLF is solely to expedite the calculation. We could do this exercise using Riemann in its full glory, though it is significantly more tedious to do so.

Expanding in the LLF, we find

$$\begin{aligned}(R_{\alpha\mu\lambda\sigma})^{\rm LLF} &= g_{\alpha\nu}\left[\partial_\lambda \Gamma^{\nu}{}_{\sigma\mu} - \partial_\sigma \Gamma^{\nu}{}_{\lambda\mu}\right] \\ &= \partial_\lambda \Gamma_{\alpha\sigma\mu} - \partial_\sigma \Gamma_{\alpha\lambda\mu} \ . \end{aligned} \tag{10.21}$$

On the second line, we have used the fact that the metric commutes with partial derivatives in the LLF. (Note also that in this form the fact that $R_{\alpha\mu\lambda\sigma} = -R_{\alpha\mu\sigma\lambda}$ is manifestly obvious.) Inserting the definition of the Christoffels and gathering terms yields

$$(R_{\alpha\mu\lambda\sigma})^{\rm LLF} = \frac{1}{2}\left(\partial_\lambda\partial_\mu g_{\alpha\sigma} - \partial_\lambda\partial_\alpha g_{\sigma\mu} - \partial_\sigma\partial_\mu g_{\alpha\lambda} + \partial_\sigma\partial_\alpha g_{\lambda\mu}\right) \ . \tag{10.22}$$

By staring at Eq. (10.22) for a little while, we notice the following symmetries:

- Antisymmetry on the last two indices: $R_{\alpha\mu\lambda\sigma} = -R_{\alpha\mu\sigma\lambda}$. We already deduced this one, but including it on this list for completeness. As already discussed, this symmetry reduces the number of independent components from n^4 to $n^3(n-1)/2$.

- Antisymmetry on the first two indices: $R_{\alpha\mu\lambda\sigma} = -R_{\mu\alpha\lambda\sigma}$. We replace the n^2 that are possible for the first two indices with $n(n-1)/2$, yielding $n^2(n-1)^2/4$. This further reduces the number of independent components to 36 for $n = 4$.

- Block symmetry of the first two with the last two: $R_{\alpha\mu\lambda\sigma} = R_{\lambda\sigma\alpha\mu}$. It's useful to think of the blocks $\alpha\mu$ and $\lambda\sigma$ as each having N possible values, with $N = n(n-1)/2$. This symmetry tells us that only $N(N+1)/2$ of them are independent; plugging in $N = n(n-1)/2$, we find that the number of components has been reduced to $n(n-1)(n^2-n+2)/8 = (n^4-2n^3+3n^2-2n)/8$, whose value is 21 when $n = 4$.

- Cyclic permutation of last three indices sums to zero: $R_{\alpha\mu\lambda\sigma} + R_{\alpha\lambda\sigma\mu} + R_{\alpha\sigma\mu\lambda} = 0$.

  - Another way to write this form is $R_{\alpha[\mu\lambda\sigma]} = 0$, where

    $$R_{\alpha[\mu\lambda\sigma]} = \frac{1}{3!}\left(R_{\alpha\mu\lambda\sigma} - R_{\alpha\lambda\mu\sigma} + R_{\alpha\lambda\sigma\mu} - R_{\alpha\mu\sigma\lambda} + R_{\alpha\sigma\mu\lambda} - R_{\alpha\sigma\lambda\mu}\right) \ , \tag{10.23}$$

    uses "total antisymmetrization" on the 3 indices μ, λ, and σ. By using the other symmetries we've established one can show that these two ways of writing this symmetry property of the Riemann tensor are completely equivalent.

  When 3 indices are completely antisymmetric, the number of independent ways of choosing them is $n(n-1)(n-2)/3!$. By folding in the other Riemann symmetries, we can in fact express this rule as $R_{[\alpha\mu\lambda\sigma]}$ (i.e., total antisymmetrization on all four indices); there are $n(n-1)(n-2)(n-3)/4! = (n^4 - 6n^3 + 11n^2 - 6n)/24$ totally antisymmetric arrangements of these indices. Bearing in that this counts up constraints on the number of independent components, we find that the number of components in the Riemann tensor is

  $$N_{\rm Riemann}(n) = \frac{n^4 - 2n^3 + 3n^2 - 2n}{8} - \frac{n^4 - 6n^3 + 11n^2 - 6n}{24} = \frac{n^2(n^2-1)}{12} \ . \tag{10.24}$$

For $n = 4$, this takes the value 20: **exactly** the number of constraints we found that we could not satisfy at second-order in displacement when we developed the idea of the local Lorentz frame.

A few other values are also interesting:

- For $n = 1$, $N_{\rm Riemann} = 0$. This is no way to develop the "curvature diamond," and thus no meaningful holonomy operation, on a 1-dimensional manifold.
- For $n = 2$, $N_{\rm Riemann} = 1$. A single number characterizes the curvature of a 2-dimensional manifold. This can be regarded as related to the radius of curvature associated with a point. For the surface of a sphere, this quantity is constant everywhere, but it varies more generally (e.g., for an oblate or prolate spheroid, or a very bumpy spheroidal body).
- For $n = 3$, $N_{\rm Riemann} = 6$. Interestingly, this is exactly the same as the number of components in a symmetric 3×3 tensor. We will revisit this observation soon.

10.4 Riemann normal coordinates

With the Riemann tensor in hand, it is possible to actually write down the metric in a freely falling representation and show how the 20 components of Riemann allow us to build the metric that is locally flat, with quadratic corrections. Those interested in a full calculation of this representation are referred to Sec. 1.11 of the textbook "A Relativist's Toolkit" by Eric Poisson, one of the suggested supplemental texts for this course.

Let γ be a timelike geodesic trajectory through spacetime, and consider a LLF around this curve. Let t be proper time on this curve, and define Cartesian coordinates x^i in this frame, undefined up to a rotation, but with $x^i = 0$ on the curve. In the LLF,

$$g_{tt} = -1 - R_{tjtk}\, x^j x^k + \mathcal{O}(x^3) \;, \tag{10.25}$$

$$g_{tj} = -\frac{2}{3} R_{tijk}\, x^i x^k + \mathcal{O}(x^3) \;, \tag{10.26}$$

$$g_{jk} = \delta_{jk} - \frac{1}{3} R_{jikl}\, x^i x^l + \mathcal{O}(x^3) \;. \tag{10.27}$$

This shows exactly how the metric components look when we perform the transformation to the local Lorentz / freely falling frame.