Massachusetts Institute of Technology
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Lecture 11
More on spacetime curvature

11.1 Variants of the curvature tensor

The Riemann curvature tensor that we derived in the previous lecture is one of the most important quantities we have developed this term. Certain variants of this curvature are also important. The first such is found by constructing the *trace*:

$$R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu} = g^{\alpha\beta} R_{\beta\mu\alpha\nu} \; . \tag{11.1}$$

This combination is known as the *Ricci* curvature tensor. Note the trace is taken on the first and third indices. A trace over the first and second indices would be identically zero, thanks to the antisymmetry of these indices; likewise a trace on the third and fourth is zero. By using the Riemann symmetries, you should be able to convince yourself that contracting on indices one and three yields the only unique trace of the Riemann curvature (up to a sign).

It is not hard to show that Ricci is symmetric on its two indices: expanding the definition of the Riemann tensor and constructing the trace yields

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\alpha\mu} + \Gamma^\alpha{}_{\alpha\beta}\Gamma^\beta{}_{\nu\mu} - \Gamma^\alpha{}_{\nu\beta}\Gamma^\beta{}_{\alpha\mu} \; . \tag{11.2}$$

Recalling the fact that the Christoffel symbol is symmetric on its bottom two indices, we see that the first and third terms on the right-hand side of (11.2) are clearly symmetric. The fourth term is also clearly symmetric, noting that the indices α and β in this term are dummies. The only unclear term is the second one. We can put it into a form that makes things much clearer by using a result from a problem set exercise:

$$\partial_\nu \Gamma^\alpha{}_{\alpha\mu} = \partial_\nu \partial_\mu \left(\ln \sqrt{|g|} \right) \; . \tag{11.3}$$

This is clearly symmetric with respect to μ and ν, and so we see that Ricci tensor is indeed symmetric.

In 4-dimensional spacetime, the Ricci tensor has 10 independent components. Heuristically, you can think of it as "half" the 4-D Riemann tensor (which we recall has 20 independent components). The other half will be developed below. Interestingly, in 3-D space, Ricci has 6 independent components — exactly the number of components that Riemann has in 3 dimensions. In 3 dimensions, Riemann and Ricci are in an important fundamental sense equivalent to one another.

The trace of the Ricci tensor,

$$R \equiv R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu} \; , \tag{11.4}$$

is called the *Ricci scalar*. It is of course just one number (or more accurately one function, since it may vary over the manifold), and totally characterizes curvature in 2 dimensions (recall that Riemann has only 1 independent component in 2-D). An interesting quantity we can construct using the Ricci scalar and the Ricci tensor is a "trace-reversed" Ricci tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \; . \tag{11.5}$$

Notice that $G^\mu{}_\mu = -R$; this can be shown using the fact that $g^{\mu\nu} g_{\mu\nu} = 4$ in 4-dimensional spacetime. We will discuss this tensor a lot more and give it a name in the very near future.

The final variant curvature tensor we will discuss is the totally trace-free part of the Riemann tensor: we take Riemann and subtract from it everything that can yield a non-zero trace. Carefully thinking about the ways in which this can be done, we end up with

$$C_{\alpha\mu\lambda\sigma} = R_{\alpha\mu\lambda\sigma} - \frac{2}{n-2}\left(g_{\alpha[\lambda}R_{\sigma]\mu} - g_{\mu[\lambda}R_{\sigma]\alpha}\right) + \frac{2}{(n-2)(n-1)}g_{\alpha[\lambda}g_{\sigma]\mu}R\,. \tag{11.6}$$

This quantity is known as the *Weyl* (pronounced like "vile" in English, named for the German mathematician Hermann Weyl) curvature tensor. The normalization of the terms that involve the metric are chosen because $g^{\mu\nu}g_{\mu\nu} = n$ in n dimensions; note that the Weyl tensor is only defined for $n \geq 3$. In the definition of $C_{\alpha\mu\lambda\sigma}$, we are using antisymmetrization on certain pairs of indices like so:

$$X_{\mu[\beta}Y_{\gamma]\nu} = \frac{1}{2}\left(X_{\mu\beta}Y_{\gamma\nu} - X_{\mu\gamma}Y_{\beta\nu}\right)\ . \tag{11.7}$$

The Weyl tensor has the same symmetries as the Riemann tensor, but thanks to the various traces that have been removed the number of independent components is equal to that of Riemann minus that of Ricci:

$$\begin{aligned} N_{\rm Weyl}(n) &= N_{\rm Riemann}(n) - N_{\rm Ricci}(n) \\ &= \frac{n^2(n^2-1)}{12} - \frac{n(n+1)}{2} \\ &= \frac{n^4 - 7n^2 - 6n}{12}\ . \end{aligned} \tag{11.8}$$

The number of independent components is 10 for $n = 4$, accounting for the "other" 10 components of Riemann that did not end up in Ricci. It is zero for $n = 3$, consistent with the fact that both Riemann and Ricci have 6 independent components in this case.

11.1.1 A brief aside on the Weyl curvature

The Weyl curvature is important for some analyses because of the simple way that it behaves under what are called *conformal transformations*. Such transformations describe a local change of scale: if two events have a proper separation (interval) of ds, then the conformal rescaling is

$$ds \to d\tilde{s} \equiv \Omega\, ds\ . \tag{11.9}$$

The function Ω, which can vary from event to event in spacetime, is called the scale factor; it is a way of introducing a remapping of spacetime that adjusts the scale associated with the separation of various events. Changing the proper separation in this way is equivalent to rescaling the metric by the conformal factor squared, allowing us to define

$$\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\ . \tag{11.10}$$

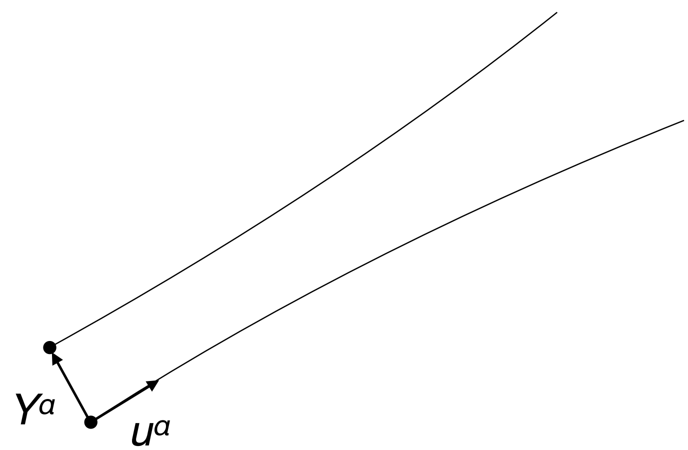
Conformal transformations of this kind are often useful because they do not affect *light cones*: if p^μ is null in the original metric, then it is also null in the conformally transformed metric: $\tilde{g}_{\mu\nu}p^\mu p^\nu = \Omega^2 g_{\mu\nu}p^\mu p^\nu = 0$. This can be particularly useful for understanding how or even whether different regions in a spacetime manifold are in causal contact with one another (i.e., if events in one part of the manifold can send a lightlike signal to events in another part of the manifold).

Unfortunately, conformal transformations tend to horribly mangle the spacetime's curvature tensors; see Appendix G of Carroll for discussion of how Riemann is affected. However, the Weyl curvature is *not* affected by such a transformation. As such, you will sometimes see Weyl called the "conformally invariant" curvature.

The Weyl curvature makes an important physical contrast with the Ricci curvature. We will soon see that the Ricci curvature is connected to the matter and energy content of spacetime. Weyl will then tell us about a spacetime's "vacuum" curvature.

11.2 Breakdown of parallelism: geodesic deviation

Our original motivating idea to lay out the notion of "curvature" was that initially parallel trajectories do not remain parallel: like lines of longitude followed from the equator (which both start on a trajectory that is 90° to the equator) to a pole, they eventually cross; or, they eventually diverge, depending on the detailed nature of the manifold's curvature. Here we make this notion precise.



11.2.1 Formal calculation

Consider a two-parameter family of geodesic trajectories in spacetime, $x^\alpha(\lambda, s)$. The parameter λ is affine parameter along each geodesic. This means that $u^\alpha(s) = (\partial x^\alpha/\partial\lambda)_s$ is the tangent along the geodesic with parameter s. (If we choose λ to be proper time, then u^α is the usual 4-velocity for this geodesic.) The parameter s tells us about the separation of neighboring geodesics. The vector[1] $Y^\alpha(\lambda) = (\partial x^\alpha/\partial s)_\lambda$ points from the event at affine parameter λ on the geodesic at s to the event at affine parameter λ on the geodesic at $s + ds$.

Before diving into our analysis, it is useful to note that

$$u^\beta \nabla_\beta Y^\mu = \frac{\partial^2 x^\mu}{\partial\lambda\partial s} + \Gamma^\mu{}_{\beta\nu} u^\beta Y^\nu \quad \text{and} \quad Y^\beta \nabla_\beta u^\mu = \frac{\partial^2 x^\mu}{\partial s\partial\lambda} + \Gamma^\mu{}_{\beta\nu} Y^\beta u^\nu \ .$$

Because partial derivatives commute, because the connection is symmetric on its downstairs indices, and because we can relabel dummy indices, we see that $u^\beta \nabla_\beta Y^\mu = Y^\beta \nabla_\beta u^\mu$. We will make use of this identity very soon.

[1] In an older version of my notes which might still be floating around, this vector was labeled ξ^α. I have changed from this older notation because $\vec{\xi}$ also denotes a Killing vector, and ξ^α will also be used in a future lecture to describe the generator of an infinitesimal coordinate transformation. The letter ξ is a bit overloaded.

Let us now evaluate the *covariant acceleration* of the separation vector Y^μ:

$$\begin{aligned}
\frac{D^2Y^\mu}{d\lambda^2} &\equiv u^\alpha\nabla_\alpha\left(u^\beta\nabla_\beta Y^\mu\right)\\
&= u^\alpha\nabla_\alpha\left(Y^\beta\nabla_\beta u^\mu\right)\\
&= \left(u^\alpha\nabla_\alpha Y^\beta\right)(\nabla_\beta u^\mu) + u^\alpha Y^\beta\nabla_\alpha\nabla_\beta u^\mu\\
&= \left(u^\alpha\nabla_\alpha Y^\beta\right)(\nabla_\beta u^\mu) + u^\alpha Y^\beta\left(\nabla_\beta\nabla_\alpha u^\mu + R^\mu{}_{\nu\alpha\beta}u^\nu\right)\\
&= \left(u^\alpha\nabla_\alpha Y^\beta\right)(\nabla_\beta u^\mu) + Y^\beta\nabla_\beta\left(u^\alpha\nabla_\alpha u^\mu\right) - \left(Y^\beta\nabla_\beta u^\alpha\right)(\nabla_\alpha u^\mu) + R^\mu{}_{\nu\alpha\beta}u^\nu u^\alpha Y^\beta\\
&= R^\mu{}_{\nu\alpha\beta}u^\nu u^\alpha Y^\beta\ .
\end{aligned}$$

The first line essentially just uses the definition $D/d\lambda \equiv u^\alpha\nabla_\alpha$ twice. To get to the second line, we use the identity we worked out above. To go to the third line, we expand the action of ∇_α. To go to the fourth line, we use the commutator rule $[\nabla_\alpha, \nabla_\beta]u^\mu = R^\mu{}_{\nu\alpha\beta}u^\nu$.

In going from the fourth to the fifth line, we use $u^\alpha\left(\nabla_\beta W_\alpha{}^\mu\right) = \nabla_\beta\left(u^\alpha W_\alpha{}^\mu\right) - \left(\nabla_\beta u^\alpha\right)W_\alpha{}^\mu$ (with $W_\alpha{}^\mu = \nabla_\alpha u^\mu$). We then simplify the fifth line by using the fact that the first and third terms on the right-hand side are equal but opposite in sign after using the identity and relabeling dummy indices. We also use the fact that, since u^μ is the tangent along a geodesic trajectory, and the quantity in parentheses in the second term is zero.

The final line gives us the covariant acceleration of the separation vector, and is known as the *equation of geodesic deviation*. Notice what it says: the separation of two nearby geodesics "accelerates" at a rate proportional to this separation and to the Riemann curvature of the spacetime in which these geodesics "live."

11.2.2 LLF frame calculation

We now look at this calculation in a somewhat less formal manner, considering how the various terms develop if we examine them in a particularly local Lorentz frame. Consider two nearby geodesics, both parameterized by affine parameter λ: Geodesic 1 follows the curve $x^\alpha(\lambda)$, and has tangent vector $u^\alpha = dx^\alpha/d\lambda$; geodesic 2 follows the curve $z^\alpha(\lambda)$, and has tangent vector $v^\alpha = dz^\alpha/d\lambda$. Let $Y^\alpha = z^\alpha - x^\alpha$ be the vector that points from events at parameter λ on geodesic 1 to events at λ on geodesic 2.

We expand all important elements of the geometry in a LLF centered on event A at $\lambda = \lambda_0$ on geodesic 1. Let A' be the corresponding event on geodesic 2. We assume that A and A' are sufficiently close to one another that their tangent vectors are parallel at $\lambda = \lambda_0$; we elaborate on what this means quantitatively below. Other important quantities we need are the metric and the connection at these two events:

$$g_{\mu\nu}\big|_A = \eta_{\mu\nu}\ , \qquad g_{\mu\nu}\big|_{A'} = \eta_{\mu\nu}\ ; \qquad \Gamma^\mu{}_{\alpha\beta}\big|_A = 0\ , \qquad \Gamma^\mu{}_{\alpha\beta}\big|_{A'} = Y^\gamma\left(\partial_\gamma\Gamma^\mu{}_{\alpha\beta}\right)\big|_A\ . \tag{11.11}$$

Notice that the connection vanishes at event A, but it does *not* vanish at A'. The value given there is the leading correction, found by Taylor-expanding over the displacement Y^α.

The geodesic equation for geodesic 1, evaluated at event A, is

$$\left.\frac{d^2x^\mu}{d\lambda^2}\right|_A = 0\ . \tag{11.12}$$

The corresponding equation for geodesic 2, evaluated at event A', is

$$\left.\frac{d^2z^\mu}{d\lambda^2}\right|_{A'} + \left(\Gamma^\mu{}_{\alpha\beta}v^\alpha v^\beta\right)\big|_{A'} = 0\ . \tag{11.13}$$

Combining our different definitions and assumptions, Eq. (11.13) can be rewritten

$$\left.\frac{d^2 z^\mu}{d\lambda^2}\right|_{A'} = -Y^\gamma \left(\partial_\gamma \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta\right)\big|_A \,. \tag{11.14}$$

Here we used $v^\alpha = u^\alpha$, which follows from our requirement that the tangents to geodesics 1 and 2 be parallel at $\lambda = \lambda_0$. This requires us to parallel transport from A' to A, but this transport is trivial[2] in the LLF.

Taking the difference between the geodesic equation on geodesic 2 and on geodesic 1, we find

$$\frac{d^2 Y^\mu}{d\lambda^2} = -\partial_\gamma \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta Y^\gamma \,. \tag{11.15}$$

We drop the A and A' subscripts from here on; all quantities are now evaluated at A.

This acceleration equation uses $d/d\lambda = u^\alpha \partial_\alpha$ as its derivative operator. To write this in proper tensorial form, we need to express things in terms of $D/d\lambda \equiv u^\alpha \nabla_\alpha$. Begin by looking at the "velocity" equation for the separation vector:

$$\frac{DY^\mu}{d\lambda} = u^\alpha \nabla_\alpha Y^\mu = \frac{dY^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta} Y^\alpha u^\beta \,. \tag{11.16}$$

It is tempting to set the connection to zero since we are in the LLF. However, we need to take one more derivative to make an acceleration equation, and so we will wait until we have taken all derivatives before zeroing our connections. The fact that the connection has non-zero slope in the LLF is important.

Taking the next derivative, we construct our acceleration for the separation vector:

$$\begin{aligned}
\frac{D^2 Y^\mu}{d\lambda^2} &= u^\gamma \nabla_\gamma \left(\frac{dY^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta} Y^\alpha u^\beta\right) \\
&= \frac{d^2 Y^\mu}{d\lambda^2} + u^\gamma \Gamma^\mu{}_{\gamma\nu} \frac{dY^\nu}{d\lambda} + \left(u^\gamma \nabla_\gamma \Gamma^\mu{}_{\alpha\beta}\right) u^\beta Y^\alpha + \Gamma^\mu{}_{\gamma\nu} \left(u^\gamma \nabla_\gamma u^\beta\right) Y^\alpha + \Gamma^\mu{}_{\alpha\beta} u^\beta \left(u^\gamma \nabla_\gamma Y^\mu\right) \\
&= \frac{d^2 Y^\mu}{d\lambda^2} + \partial_\gamma \Gamma^\mu{}_{\alpha\beta} u^\beta u^\gamma Y^\alpha + O(\Gamma^2) \,.
\end{aligned} \tag{11.17}$$

On the final line of this equation, we have gone into the LLF near event A, we have used the fact that u^α is a geodesic, and we used the fact that $dY^\mu/d\lambda = v^\mu - u^\mu = 0$.

Finally, use the previous result we found for $d^2 Y^\mu/d\lambda^2$ to write this

$$\begin{aligned}
\frac{D^2 Y^\mu}{d\lambda^2} &= \partial_\gamma \Gamma^\mu{}_{\alpha\beta} u^\beta u^\gamma Y^\alpha - \partial_\gamma \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta Y^\gamma \\
&= \left(\partial_\gamma \Gamma^\mu{}_{\alpha\beta} - \partial_\alpha \Gamma^\mu{}_{\beta\gamma}\right) u^\beta u^\gamma Y^\alpha \\
&= R^\mu{}_{\beta\gamma\alpha} u^\beta u^\gamma Y^\alpha \,.
\end{aligned} \tag{11.18}$$

In going from the first to the second line, we relabeled dummy indices; to go to the third line, we identified that combination of connection derivatives as the Riemann curvature in the LLF. The final result is identical to that found in the formal calculation, modulo a few different choices of dummy index labels.

This second version of the calculation highlights how it is that the second order terms in the metric (via the first order correction to the connection) cause nearby geodesics to deviate from one another. This is the hallmark of a *tidal* effect: the geodesics deviate because "free fall" means slightly different things to freely falling observers that are slightly separated from one another.

[2] At least to linear order in important quantities; if you do this transport and keep all terms, you will find an additional term of order Γ^2. We discard this term at the end of the calculation by our assumption that we are in the LLF.

11.3 A differential curvature rule: The Bianchi identity

Our final identity is one involving derivatives of the Riemann tensor. To derive it, recall the identity involving the commutator of covariant derivatives:

$$[\nabla_\lambda, \nabla_\sigma] p_\alpha = -R^\mu{}_{\alpha\lambda\sigma} p_\mu \ . \tag{11.19}$$

It is not hard to show that this generalizes to a higher rank tensor:

$$[\nabla_\lambda, \nabla_\sigma] M_{\alpha\beta} = -R^\mu{}_{\alpha\lambda\sigma} M_{\mu\beta} - R^\mu{}_{\beta\lambda\sigma} M_{\alpha\mu} \ . \tag{11.20}$$

With this established, let us examine two relations. The first is

$$\boxed{[\nabla_\alpha, \nabla_\beta] \nabla_\gamma p_\delta = -R^\mu{}_{\gamma\alpha\beta} \nabla_\mu p_\delta - R^\mu{}_{\delta\alpha\beta} \nabla_\gamma p_\mu} \tag{11.21}$$

The second requires some manipulation:

$$\begin{aligned} \nabla_\alpha [\nabla_\beta, \nabla_\gamma] p_\delta &= \nabla_\alpha \left(-R^\mu{}_{\delta\beta\gamma} p_\mu\right) \\ &= -p_\mu \nabla_\alpha R^\mu{}_{\delta\beta\gamma} - R^\mu{}_{\delta\beta\gamma} \nabla_\alpha p_\mu \\ &= -p^\mu \nabla_\alpha R_{\mu\delta\beta\gamma} - R^\mu{}_{\delta\beta\gamma} \nabla_\alpha p_\mu \ . \end{aligned} \tag{11.22}$$

On the final line, used the fact that the metric commutes with the covariant derivative to raise the index μ on the 1-form component and simultaneously lower it on the Riemann curvature. We finally use Riemann symmetry to swap some of the indices around on the first term on the right-hand side to write this equation

$$\boxed{\nabla_\alpha [\nabla_\beta, \nabla_\gamma] p_\delta = -p^\mu \nabla_\alpha R_{\beta\gamma\mu\delta} - R^\mu{}_{\delta\beta\gamma} \nabla_\alpha p_\mu} \tag{11.23}$$

Now, let's look at what happens when we antisymmetrize these equations on the indices α, β, and γ. Begin with the left-hand side of Eq. (11.20):

$$\begin{aligned} [\nabla_{[\alpha}, \nabla_\beta] \nabla_{\gamma]} p_\delta &= \frac{1}{3!} \Big([\nabla_\alpha, \nabla_\beta] \nabla_\gamma + [\nabla_\beta, \nabla_\gamma] \nabla_\alpha + [\nabla_\gamma, \nabla_\alpha] \nabla_\beta \\ &\qquad - [\nabla_\alpha, \nabla_\gamma] \nabla_\beta - [\nabla_\gamma, \nabla_\beta] \nabla_\alpha - [\nabla_\beta, \nabla_\alpha] \nabla_\gamma \Big) p_\delta \ . \end{aligned} \tag{11.24}$$

Expanding every one of these commutators, one finds that this can be written equivalently as

$$\begin{aligned} [\nabla_{[\alpha}, \nabla_\beta] \nabla_{\gamma]} p_\delta &= \frac{1}{3!} \Big(\nabla_\alpha [\nabla_\beta, \nabla_\gamma] + \nabla_\beta [\nabla_\gamma, \nabla_\alpha] + \nabla_\gamma [\nabla_\alpha, \nabla_\beta] \\ &\qquad - \nabla_\alpha [\nabla_\gamma, \nabla_\beta] - \nabla_\gamma [\nabla_\beta, \nabla_\alpha] - \nabla_\beta [\nabla_\alpha, \nabla_\gamma] \Big) p_\delta \\ &= \nabla_{[\alpha} [\nabla_\beta, \nabla_{\gamma]}] p_\delta \ . \end{aligned} \tag{11.25}$$

This is exactly the left-hand side of Eq. (11.22), antisymmetrized on these three indices. In other words, **antisymmetrizing the left-hand side of Eq. (11.20) on α, β, and γ is exactly the same as antisymmetrizing the left-hand side of Eq. (11.22).**

If this antisymmetrization makes the left-hand sides equal, then it must make the right-hand sides equal too. Let's look at what we get when we antisymmetrize the right-hand sides of (11.20) and (11.22) on these three indices and equate them to each other:

$$R^{\mu}{}_{[\gamma\alpha\beta]}\nabla_{\mu}p_{\delta} + R^{\mu}{}_{\delta[\alpha\beta}\nabla_{\gamma]}p_{\mu} = p^{\mu}\nabla_{[\alpha}R_{\beta\gamma]\mu\delta} + R^{\mu}{}_{\delta[\beta\gamma}\nabla_{\alpha]}p_{\mu}\ . \tag{11.26}$$

The first term on the right-hand side of (11.25) vanishes by virtue of the Riemann symmetry $R^{\mu}{}_{[\gamma\alpha\beta]} = 0$. The second term on the right-hand side of (11.25) is identical to the second term on the left-hand side of (11.25); some of the indices are ordered slightly differently, but because they are being antisymmetrized, we can cyclically permute them without changing the equation. These two terms thus cancel out. What remains is

$$p^{\mu}\nabla_{[\alpha}R_{\beta\gamma]\mu\delta} = 0\ . \tag{11.27}$$

This must hold for any p^{μ}, so we conclude that

$$\boxed{\boxed{\nabla_{[\alpha}R_{\beta\gamma]\mu\delta} = 0}} \tag{11.28}$$

Using the antisymmetry on the first two indices, this can be written in the equivalent form

$$\boxed{\boxed{\nabla_{\alpha}R_{\beta\gamma\mu\delta} + \nabla_{\beta}R_{\gamma\alpha\mu\delta} + \nabla_{\gamma}R_{\alpha\beta\mu\delta} = 0}} \tag{11.29}$$

Equations (11.27) and (11.28) are both known as the *Bianchi identity.* The Bianchi identity is the last major ingredient that we need to turn our theory of spacetime curvature into a fully fledged theory of gravity.