Variants of the curvature tensor:

1. Take the trace on the $1^{st} + 3^{rd}$ indices:

$$R^\alpha_{\mu \nu} = g^{\alpha \beta} R_{\beta \mu \nu} = R_{\mu \nu}$$

The Ricci curvature tensor.

Easy to show this is symmetric on $\mu \leftrightarrow \nu$:

$$R_{\mu \nu} = \partial_\nu R^\alpha_{\mu \alpha} - \partial_\mu R^\alpha_{\nu \alpha} + R^\gamma_\mu R^\beta_{\nu \gamma} - R^\gamma_\nu R^\beta_{\mu \gamma}$$

$$\Rightarrow (\ln |g|^{1/2})$$

$\Rightarrow$ 10 components in 4D - sort of like "half" of the Riemann tensor.

Trace on any other pair of indices is either zero or related using symmetries.

1'. Trace of Ricci:

$$g^{\mu \nu} R_{\mu \nu} = R = R^\alpha_{\alpha \mu}$$

"Ricci scalar" or "curvature scalar".

Totally describes curvature in 2-D.
2. Ricci is "trace Riemann" --- "rest" of Riemann is the trace-free part; the Weyl curvature.

\[ C_{\mu\nu} = R_{\mu\nu} - \frac{2}{n-2} \left( g_{\mu\nu} R - g_{\mu\nu} R \right) + \frac{2}{(n-2)(n-1)} g_{\mu\nu} R \]

- Only defined for \( n \geq 3 \)
- Has the same symmetries as Riemann, but designed such that trace on any pair of indices is zero.

Number of independent components:

\[ N_{\text{Weyl}} = \frac{n^2 (n^2 - 1)}{12} - \frac{n(n+1)}{2} \]

\[ = n \left( n^3 - 7n - 6 \right) \]

\[ N_{\text{Weyl}}(3D) = 0 \]

\[ N_{\text{Weyl}}(4D) = 10 \]
Trick for Weyl: Conformal transformation

Conformal transformations are a local change of scale:
\[ ds \to \Omega \, ds = d\tilde{s} \]
\[ \Rightarrow \text{ Scale factor: function of the spacetime.} \]

Shows up in metric via
\[ \tilde{g}^{\mu\nu} = \Omega^2(x) \, g^{\mu\nu} \]

Often used because it leaves "light cones" invariant: if \( p^\mu \) is null in the original metric, \( g^{\mu\nu} p_\mu p_\nu = 0 \), then
\[ \tilde{g}^{\mu\nu} p_\mu p_\nu = \Omega^2 g^{\mu\nu} p_\mu p_\nu = 0. \]

Curvature tensor (Riemann) gets horribly mangled by conformal transformation (see Carroll, Appendix G) ... but Weyl is invariant! Sometimes see Weyl called conformally invariant component of the curvature.

More physical (for us): Will soon see that Ricci is related to matter-energy content of spacetime; Weyl will then describe a fundamental "vacuum" curvature.
Breakdown of parallelism: Initially parallel trajectories (geodesics) become non-parallel. The geodesics deviate from their initial parallelism. (Yucky discussion: Carroll pp 145-146)

Consider two nearby geodesics, each parameterized by affine parameter $\lambda$:

\[ A \rightarrow \lambda_0 \quad \vec{\gamma}_V, \text{ tangent vector} \vec{v} \]

\[ A \rightarrow \lambda_0 \quad \vec{\gamma}_U, \text{ tangent vector} \vec{u} \]

Points from events at a particular $\lambda$ on $\vec{\gamma}_U$ to events at that same $\lambda$ on the other, $\vec{\gamma}_V$:

\[ \delta \vec{y} = \vec{x} (\vec{\gamma}_V, \lambda) - \vec{x} (\vec{\gamma}_U, \lambda) \]

Finally, assume the curves begin parallel:

\[ \vec{u} (\lambda_0) = \vec{v} (\lambda_0) \quad (\text{Curves must be close enough that they have initial tangent space to 1st order}) \]

This implies \( (u^\beta \partial_{\beta}) \big|_{\lambda_0} = 0 \)

\[ \Rightarrow \text{use as a boundary condition.} \]
Goal now: develop an "acceleration equation" for $\ddot{x}^\gamma$ by comparing geodesics along $\gamma_1$ to that along $\gamma_2$.

Make analysis simple & bring in some intuition by starting in a LCF constructed centered on A. Not necessary!

Just nice to clean things up.

Geodesic equation for $\gamma_1$ at $A$: \[ \frac{d^2x^\gamma}{dx^2} \bigg|_{A} = 0 \quad \text{(no } \Gamma \text{'s in LCF)} \]

Geodesic equation for $\gamma_2$ at $A'$: \[ \frac{d^2x^\gamma}{dx^2} \bigg|_{A'} + \left[ \Gamma^\gamma{}_{\mu\nu} \frac{dx^\mu}{dx} \frac{dx^\nu}{dx} \right]_{A'} = 0 \]

\[ \Gamma^\gamma{}_{\mu\nu} \bigg|_{A'} = (\partial_\beta \Gamma^\gamma{}_{\mu\nu})_{A} \delta^\beta \]

\[ \frac{dx^\mu}{dx} \bigg|_{A'} = V^\mu = u^\mu \quad \text{initially parallel} \]

\[ \Rightarrow \frac{d^2x^\gamma}{dx^2} \bigg|_{A'} = -\partial_\beta \Gamma^\gamma{}_{\mu\nu} u^\mu u^\nu \delta^\beta \]

Difference between geodesic equations gives us an acceleration of the difference:

\[ \frac{d^2x^\gamma}{dx^2} \bigg|_{A'} - \frac{d^2x^\gamma}{dx^2} \bigg|_{A} = \frac{d^2\ddot{x}^\gamma}{dx^2} = -\partial_\beta \Gamma^\gamma{}_{\mu\nu} u^\mu u^\nu \delta^\beta \]
Not bad! However, not tensorial. E.g., \( \frac{d}{dx} = u^x \cdot \partial_x \) - want to re-express as much as possible using covariant derivatives
try to get something that will be invariant in all reference frames.

Re-express using covariant derivative trajectory:

\[
\frac{D^2 \xi^d}{dx^2} = u^\beta \partial_\beta \xi^d = u^\beta \partial_\beta \xi^d + u^\beta \Gamma^d_{\beta \mu} \xi^\mu
\]

\[
= \frac{d^2 \xi^d}{dx^2} + \Gamma^d_{\beta \mu} u^\beta \xi^\mu \quad \text{Don't zero yet \((\text{UF})\) -}
\]

\[
\quad \text{wait til we've done another derivative.}
\]

\[
\frac{D^2 \xi^d}{dx^2} = u^\gamma \nabla_\gamma \left( \frac{d \xi^d}{dx} + \Gamma^d_{\beta \mu} u^\beta \xi^\mu \right)
\]

1st two terms:
\( u^\gamma \nabla_\gamma \) applied to
1st term in paren.
Last three terms:
\( u^\gamma \nabla_\gamma \) applied to \( \sum_0 \) term in paren.

\[
\rightarrow \frac{D^2 \xi^d}{dx^2} = \frac{d^2 \xi^d}{dx^2} + \partial_\gamma \left( u^\gamma \nabla_\gamma \right) + \Gamma^d_{\beta \mu} u^\beta \xi^\mu \xi^\mu + O(p^2)
\]

\( \text{toss in \(\text{UF}\).} \)
Combine with result for $\frac{d^2 \zeta}{d\lambda^2}$:

$$\frac{d^2 \zeta}{d\lambda^2} = \partial_\beta \Gamma^\nu_{\beta \mu} u^\mu v^\nu - \partial_\mu \Gamma^\nu_{\beta \nu} u^\mu v^\beta$$

Relabel dummy: $\beta \rightarrow \mu$, $\mu \rightarrow \nu$, $\nu \rightarrow \beta$ on 2nd term:

$$\rightarrow \frac{d^2 \zeta}{d\lambda^2} = (\partial_\beta \Gamma^\nu_{\beta \mu} - \partial_\mu \Gamma^\nu_{\beta \nu}) u^\mu v^\nu$$

$$\frac{D^2 \zeta}{d\lambda^2} = R^\nu_{\rho \mu} u^\rho v^\mu$$

Equation of geodesic deviation: gives vs a covariant - tensorial! - notion of the action of tidal.
Recall Riemann is the action of the commutator of two derivatives: 
\[ [\nabla_x, \nabla_y] \nabla^a = R^{a}_{\, \mu \nu \sigma} \nabla^\mu \nabla^\nu \nabla^\sigma \]

- Equivalent to our holonomic definition.

Generalized: 
\[ [\nabla_x, \nabla_y] F^\rho = R^{a}_{\, \mu \nu \rho} F^{\mu \nu} - R^{a}_{\, \rho \nu \sigma} F^{\rho \sigma} \]

Consider the following two relations:

A \[ [\nabla_x, \nabla_y] \nabla^a \rho \delta = -R^{\alpha}_{\, \gamma \mu \nu} \nabla^\gamma \rho \delta - R^{\gamma}_{\, \delta \mu \nu} \nabla^\delta \rho \mu \]

B \[ \nabla_x [\nabla_y, \nabla_z] \rho \delta = \nabla_x (-R^{\alpha}_{\, \delta \mu \nu} \rho \mu) \]

metric components \( \omega / \nabla \)

= \[ -\rho \mu \nabla_x R^{\mu \nu \rho \delta} - R^{\mu \nu \rho \delta} \nabla_x \rho \mu \]

Riemann geometry.

= \[ -\rho \mu \nabla_x R^{\mu \nu \rho \delta} \rho \delta - R^{\mu \nu \rho \delta} \nabla_x \rho \mu \]

Now, antisymmetrize on \( \alpha, \beta, \delta \). Consider LHS of A:

\[ [\nabla_x, \nabla_y] \nabla^a \rho \delta = \frac{1}{3!} \left( [\nabla_x, \nabla_y] \nabla^a \rho \delta + [\nabla_y, \nabla_x] \nabla^a \rho \delta + [\nabla_x, \nabla_y] \nabla^a \rho \delta \right) \]

= \[ [\nabla_x, \nabla_y] \nabla^a \rho \delta - [\nabla_y, \nabla_x] \nabla^a \rho \delta - [\nabla_x, \nabla_y] \nabla^a \rho \delta \]

= \[ \frac{1}{3!} \left( \nabla_x [\nabla_y, \nabla_x] + \nabla_y [\nabla_x, \nabla_y] + \nabla_x [\nabla_y, \nabla_x] \right) \rho \delta \]

= \[ \nabla_x [\nabla_y, \nabla_x] \rho \delta \]

= \[ LHS \ of \ B, \ \text{antisymmetrized} \]
Antisymmetrization makes the equations equal! Let's examine Riemann:

\[ R^m_\ [\epsilon \rho \sigma] \nabla_\mu \rho \sigma + R^m_\ [\epsilon \rho \sigma \nabla_\nu] \rho \mu \rightarrow \text{the same} \]

\[ = \rho^m \nabla_{[\epsilon R_{\beta \gamma}] \mu \delta} + R^m_\ [\beta \gamma] \nabla_\delta \rho \mu \]

Zero by Riemann symmetry.

\[ \rightarrow \rho^m \nabla_{[\epsilon R_{\beta \gamma}] \mu \delta} = 0 \]

Holds for any \( \rho^m \); so, we find the Bianchi identity:

\[ \nabla_{[\epsilon R_{\beta \gamma}] \mu \delta} = 0 \]

\[ \nabla_\epsilon R_{\beta \gamma} \mu \delta + \nabla_\beta R_{\gamma \epsilon \mu \delta} + \nabla_\gamma R_{\epsilon \beta \mu \delta} = 0 \]

Rock on.