

MASSACHUSETTS INSTITUTE OF TECHNOLOGY  
DEPARTMENT OF PHYSICS  
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LECTURE 12  
MAKING A THEORY OF GRAVITY

### 12.1 Final prep: Revisiting the Bianchi identity

In the previous lecture, we derived the Bianchi identity for the Riemann tensor:

$$\nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} + \nabla_\gamma R_{\alpha\beta\mu\nu} = 0. \quad (12.1)$$

Let's hit this equation with  $g^{\beta\mu}$ . Because the metric has no covariant derivative, it commutes with all of the derivatives, and we find the following variant of this identity:

$$\nabla_\alpha R_{\gamma\nu} + \nabla^\mu R_{\gamma\alpha\mu\nu} - \nabla_\gamma R_{\alpha\nu} = 0. \quad (12.2)$$

(Why the minus sign on the final term? The contraction is on indices 2 and 3, rather than indices 1 and 3 for the “standard” contraction from Riemann to Ricci. We thus get a minus sign from the associated Riemann symmetry.) Let's do this again, now using  $g^{\gamma\nu}$ :

$$\nabla_\alpha R - \nabla^\mu R_{\alpha\mu} - \nabla^\nu R_{\alpha\nu} = 0. \quad (12.3)$$

Recognizing that  $\mu$  and  $\nu$  are dummy indices in those two terms and that  $\nabla_\alpha = \nabla^\mu g_{\alpha\mu}$ , we can reorganize this as

$$\nabla^\mu \left( R_{\alpha\mu} - \frac{1}{2} g_{\alpha\mu} R \right) = 0. \quad (12.4)$$

The quantity in parenthesis is the as-yet-unnamed tensor  $G_{\alpha\mu}$  which we briefly noted in the previous lecture as the “trace-reversed” Ricci curvature. We now see that it has a further interesting property, namely that its divergence vanishes:

$$\nabla^\mu G_{\mu\alpha} = 0. \quad (12.5)$$

(Using its symmetry to swap indices around.) Divergence-free quantities tend to show up when we connect a form of mathematical analysis to physics. In anticipation of the possibility that this tensor might prove useful for us, we will call it the *Einstein curvature* tensor.

### 12.2 Making a theory of gravity: Guiding principles

We are now ready to put all of the ingredients we have developed over the previous 11 lectures together to make a relativistic theory of gravity. Two main ingredients go into this: first, using the principle of equivalence (in the form of the Einstein equivalence principle) to express laws of physics (particularly laws of motion) in a form appropriate to physics in a curved spacetime; and second, some sort of field equation that connects spacetime to sources of matter and energy.

We have already had some practice with using the equivalence principle. The key idea, sometimes called the “minimal coupling<sup>1</sup>” principle, is to begin with a law of physics that is valid in inertial coordinates in flat spacetime; to then use that form of the law in a local Lorentz frame in which, by the Einstein equivalence principle, it should hold as well; and then to rewrite the law in a covariant, tensorial form. We finally assert that this law holds in curved spacetime. An example

<sup>1</sup>In contrast to “curvature coupling,” which can modify these laws when the physical process under discussion extends over a large enough region that processes couple to the spacetime's curvature tensors.

of this is how we describe force-free motion: in a freely falling frame, bodies that move unaffected by non-gravitational forces follow a trajectory describe by

$$\frac{d^2 x^\mu}{d\tau^2} = 0, \quad (12.6)$$

a “straight” line in the inertial coordinates. A more general statement of such motion is that the tangent vector is parallel transported along the trajectory, so the fully covariant formulation of becomes

$$u^\alpha \nabla_\alpha u^\beta = 0. \quad (12.7)$$

Equations (12.6) and (12.7) are identical in a LLF. However, the second form is written in a fully tensorial, covariant form, so we assert that it describes motion more generally.

A second example, which will be quite relevant very shortly, is our law of local energy and momentum conservation. In special relativity and in inertial coordinates, we write this law

$$\partial_\mu T^{\mu\nu} = 0. \quad (12.8)$$

To make this tensorial, we simply need to “upgrade” the derivative from partial to covariant:

$$\nabla_\mu T^{\mu\nu} = 0. \quad (12.9)$$

### 12.3 The Newtonian limit

A principle we must hold in mind over all of these ideas is that the theory of gravity we develop must be compatible with a Newtonian limit. Kepler’s laws, for example, are extremely well tested, and empirically describe the motion of bodies in our solar system to extraordinarily good accuracy. At least for non-relativistic motion, we know that Newtonian gravity works very well; our relativistic theory of gravity must incorporate this limit.

The equation of motion in Newtonian gravity is

$$\frac{d^2 x^i}{dt^2} = -\nabla\Phi = -\delta^{ij} \partial_j \Phi. \quad (12.10)$$

This is very much *not* a covariant formulation (a specific time  $t$  and spatial coordinates  $x^i$  have been picked out), but our covariant equation of motion should reduce to this in an appropriate limit. We begin with the geodesic equation, which we write as a second-order equation for the trajectory  $x^\mu(\tau)$  that a freely falling body follows:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \quad (12.11)$$

The slow motion limit is defined by requiring that the timelike component of the 4-velocity be much larger than the spatial components:

$$\frac{dx^0}{d\tau} = \frac{dt}{d\tau} \gg \frac{dx^i}{d\tau} \quad \text{for all } i. \quad (12.12)$$

(You should be able to convince yourself that this is equivalent to  $v^i \ll c$  for all  $i$ .) With this simplification, we expect that the  $\alpha = 0, \beta = 0$  indices contribute much more to the sum over dummy indices than any of the others and we write this

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{00} \left( \frac{dt}{d\tau} \right)^2 = 0 \quad \text{or} \quad \frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{00} \left( \frac{dt}{d\tau} \right)^2, \quad (12.13)$$

moving the Christoffel term to the right-hand side in anticipation of how we wish to present this result later.

The Christoffel symbol we must now compute is

$$\Gamma^\mu{}_{00} = \frac{1}{2} g^{\mu\nu} (\partial_t g_{\nu 0} + \partial_t g_{0\nu} - \partial_\nu g_{00}) . \quad (12.14)$$

To recover the Newtonian limit, we can take the metric to be time independent, yielding

$$\Gamma^\mu{}_{00} = -\frac{1}{2} g^{\mu\nu} \partial_\nu g_{00} . \quad (12.15)$$

We also imagine that the spacetime metric is *almost* that of special relativity, writing

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} , \quad (12.16)$$

where all of the components of  $h_{\mu\nu}$  are “small”: all of its components are much less than 1. It’s not hard to show that the inverse tensor corresponding to (12.16) is given by

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \mathcal{O}(h^2) , \quad (12.17)$$

where  $h^{\mu\nu} = \eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}$ . Since all the components of  $h_{\mu\nu}$  are considered to be small, we will drop contributions  $\mathcal{O}(h^2)$ ; this is why we raise indices on  $h_{\mu\nu}$  with  $\eta^{\mu\nu}$ . Putting all this together yields for our Christoffel symbol

$$\Gamma^\mu{}_{00} = -\frac{1}{2} \eta^{\mu\nu} \partial_\nu h_{00} . \quad (12.18)$$

Our equation of motion then becomes

$$\frac{d^2 t}{d\tau^2} = 0 \quad (\text{using } \partial_t h_{00} = 0) , \quad (12.19)$$

$$\frac{d^2 x^i}{d\tau^2} = \frac{1}{2} \delta^{ij} \partial_j h_{00} \left( \frac{dt}{d\tau} \right)^2 \quad \text{or ,}$$

$$\frac{d^2 x^i}{dt^2} = \frac{1}{2} \delta^{ij} \partial_j h_{00} . \quad (12.20)$$

The equation for  $t$  tells us that in this limit a freely falling observer’s proper time is the same as coordinate time, up to a linear transformation (i.e., change of origin, change of units). The equation governing  $x^i$  is the gravitational Newtonian equation of motion if

$$h_{00} = -2\Phi \quad \text{or} \quad g_{00} = -(1 + 2\Phi) . \quad (12.21)$$

We will hold this correspondence with the Newtonian limit in our heads as we develop a field equation for spacetime.

## 12.4 A field equation for relativistic gravity

The next, and arguably most crucial step of all, is to develop a field equation for spacetime: some kind of relation that tells us how much “gravity” arises per unit of “source.” Based on what we’ve developed so far, our “gravity” must be upgraded from a Newtonian gravitational potential to something like the spacetime metric. But we need to think carefully about what to use for “source” in such a relationship.

As guidance, let’s write down the field equation of Newtonian gravity:

$$\nabla^2 \Phi = \delta^{ij} \partial_i \partial_j \Phi = 4\pi G \rho . \quad (12.22)$$

There is a lot flawed about this equation from a relativistic perspective. Note that it is manifestly not covariant or tensorial: the right-hand side is  $\rho$ , which in Newtonian physics we have always taken to be mass density. In our  $c = 1$  units, this is the same thing as energy density. Energy density, however, we now recognize as not a scalar, but rather as one component of the stress energy tensor:  $\rho = T_{00}$ . To make this tensorial, one hypothesis is that we should “upgrade” the right-hand side from something proportional to  $\rho$  to something proportional to  $T_{\mu\nu}$ .

On the left-hand side, we have spatial derivatives (not covariant: we’ve chosen time and space) acting on the Newtonian potential  $\Phi$ . Our analysis of Newtonian motion shows us that the metric and  $\Phi$  play similar roles as far as gravitational motion is concerned, so presumably we want something involving two derivatives of the metric. Heuristically, our proposed field equation has the structure

$$\text{(two derivatives of the spacetime metric)} = T_{\mu\nu} . \quad (12.23)$$

Two derivatives of the spacetime metric is what goes into making curvature tensors (along with a few nonlinear combinations with the metric, and of first derivatives of the metric). So a plausible hypothesis for the left-hand side is one of the curvature tensors that we developed. Clearly it cannot be the Riemann tensor — Riemann has too many indices, and too many independent components to equate to  $T_{\mu\nu}$ . Both Ricci and Einstein, however, fit. To choose which one, recall that the stress-energy tensor is divergence free:  $\nabla^\mu T_{\mu\nu} = 0$ . The tensor which goes on the left-hand side must be divergence free as well; this picks out the Einstein tensor as what we should put on the left-hand side:

$$G_{\mu\nu} = \kappa T_{\mu\nu} . \quad (12.24)$$

We have introduced a constant of proportionality  $\kappa$  because curvature and stress-energy have different dimensions; there must be some factor to have sure that their units are consistent. Our last task is to fix this constant.

One bit of sleight of hand helps to organize terms for this analysis. Recall that  $G_{\mu\nu}$  is the “trace-reversed” Ricci tensor; you should be able to convince yourself that if you trace-reverse  $G_{\mu\nu}$  you just get the Ricci tensor  $R_{\mu\nu}$  back. We can trace-reverse any 2-index tensor by subtracting one half the metric times the trace of the tensor. Taking the trace reverse of both sides of Eq. (12.24) gives us a second way of writing our proposed field equation which will prove handy in the next calculation:

$$R_{\mu\nu} = \kappa \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) , \quad \text{where} \quad T = g^{\mu\nu} T_{\mu\nu} . \quad (12.25)$$

It should be emphasized that Eqs. (12.24) and (12.25) are essentially identical, differing only by a trace-reverse operation.

In order to fix the constant  $\kappa$ , we take everything to be static, and go to the weak-field limit in a particular coordinate representation which allows us to write  $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ . The components of  $h_{\mu\nu}$  are all taken to be small, meaning that we will discard any term of  $\mathcal{O}(h^2)$ . Let’s take our source to be a static perfect fluid:

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \simeq \rho u_\mu u_\nu . \quad (12.26)$$

We are using the idea that for a non-relativistic fluid,  $\rho \gg P$ . Bearing in mind that  $\partial P / \partial \rho = c_s^2$ , where  $c_s$  is the speed of sound, this is akin to saying that the speed of sound is much smaller than the speed of light, a reasonable condition to impose for the Newtonian limit.

Requiring the fluid to be static implies that  $u^\mu \doteq (u^0, 0, 0, 0)$ . To normalize the only non-zero velocity component, we enforce

$$g_{\mu\nu} u^\mu u^\nu = -1 \quad \longrightarrow \quad u^0 = 1 + \frac{1}{2} h_{00} + \mathcal{O}(h^2) . \quad (12.27)$$

Lowering the index so that we have the component we need in our stress-energy tensor yields

$$u_0 = g_{0\mu}u^\mu = (\eta_{00} + h_{00})u^0 = -1 + \frac{1}{2}h_{00} + \mathcal{O}(h^2) . \quad (12.28)$$

We now have enough pieces to generate at least one component of the proposed field equation: let's look at

$$R_{00} = \kappa \left( T_{00} - \frac{1}{2}g_{00}T \right) . \quad (12.29)$$

On the right-hand side, discarding terms of  $\mathcal{O}(h^2)$ , we have

$$T_{00} = \rho u_0 u_0 = \rho(1 - h_{00}) , \quad (12.30)$$

$$T = g^{\mu\nu}T_{\mu\nu} = \rho u^\mu u_\mu = -\rho , \quad (12.31)$$

so we find

$$T_{00} - \frac{1}{2}g_{\mu\nu}T = \frac{1}{2}\rho + \mathcal{O}(h) . \quad (12.32)$$

As we'll see, it is acceptable to truncate at this order on the right-hand side.

Let's now build the left-hand side:

$$R_{00} = R^\alpha{}_{0\alpha 0} = R^i{}_{0i0} . \quad (12.33)$$

We can reduce the sum over spacetime index  $\alpha$  to a sum over spatial index  $i$  since  $R^0{}_{000}$  is zero by Riemann symmetries. Inserting the definition of Riemann in terms of Christoffel derivatives and nonlinear Christoffel terms,

$$\begin{aligned} R_{00} &= \partial_i \Gamma^i{}_{00} - \partial_0 \Gamma^i{}_{0i} + \mathcal{O}(\Gamma^2) \\ &= \frac{1}{2} \partial_i [g^{i\mu} (\partial_0 g_{\mu 0} + \partial_0 g_{0\mu} - \partial_\mu g_{00})] \\ &= -\frac{1}{2} \partial_i [\eta^{i\mu} \partial_\mu h_{00}] \\ &= -\frac{1}{2} \delta^{ij} \partial_i \partial_j h_{00} \\ &\equiv -\frac{1}{2} \nabla^2 h_{00} . \end{aligned} \quad (12.34)$$

In each step, we simplify by discarding terms that involve  $\partial_0$  (nothing depends on time), by using the fact that derivatives of  $g_{\mu\nu}$  become derivatives of  $h_{\mu\nu}$  since  $\eta_{\mu\nu}$  is a constant, and by keeping only terms that are linear in  $h$ .

Equating  $R_{00}$  to  $T_{00} - (1/2)g_{00}T$ , we find

$$\nabla^2 h_{00} = -\kappa \rho . \quad (12.35)$$

Recovering Newtonian free-fall motion in the limit requires that  $h_{00} = -2\Phi$ . Making this substitution, we see that this limit of our proposed field equation duplicates the Newtonian field equation if we put  $\kappa = 8\pi G$ . This allows us to at last write down a field equation for spacetime:

$$\boxed{G_{\mu\nu} = 8\pi G T_{\mu\nu}} \quad (12.36)$$

This is known as the *Einstein field equation*, often abbreviated as just “the Einstein equation” or “the Einstein equations,” depending on whether you think of this as one tensor equation with 10 components, or 10 coupled equations<sup>2</sup>.

<sup>2</sup>Note that if we had kept the  $\mathcal{O}(h)$  term in Eq. (12.32), it would have changed the differential equation in such a way as to generate an  $\mathcal{O}(h^2)$  correction to the solution. This is ignorable in a leading-order analysis; we will revisit issues like this when we look at solutions to this equation in more detail.

## 12.5 A few remarks on the Einstein field equation

The way we have derived the Einstein field equation here parallels how Einstein originally derived it: deduce that the source should be  $T_{\mu\nu}$ , infer that the metric plays the role of a potential in relativistic theory, and then use a divergence-free curvature as the quantity which is produced by this source. This is a rather ad hoc derivation. You might wonder — why do we make these particular choices? Could we not come up with other ways of combining tensors that is divergence free and covariant?

Indeed we can. In fact, one such modification is not hard to notice. Another divergence-free tensor is the metric itself (since *any* covariant derivative of the metric is zero). The metric has the wrong dimensions or units to be curvature, but perhaps we could introduce some scalar constant that corrects this. The result is a field equation of the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} . \quad (12.37)$$

The constant scalar factor  $\Lambda$  introduced here is known as the *cosmological constant*. It's actually a matter of taste whether this term belongs on the left-hand side or the right-hand side of this field equation. On the left-hand side, it introduces a kind of mean curvature everywhere in spacetime. Moving it to the right-hand side, we can interpret it as a kind of stress-energy tensor:

$$T_{\mu\nu}^{\Lambda} = -\frac{\Lambda}{8\pi G} g_{\mu\nu} . \quad (12.38)$$

This can be thought of as a rather bizarre perfect fluid, one that has

$$\rho = \frac{\Lambda}{8\pi g} , \quad P = -\rho \quad (12.39)$$

in a freely-falling frame. Such a stress-energy tensor arises in quantum field theory, and represents a form of vacuum energy, as it is isotropic and invariant to Lorentz transformations in the local Lorentz frame. We will set  $\Lambda = 0$  in most of our analysis, though nonzero  $\Lambda$  is especially important and interesting in cosmology.

In the next lecture, we will re-derive (12.36) by working from an action principle. In doing so, we will see that Eq. (12.36) is, in a very well-defined way, the *simplest* theory of gravity that is compatible with general covariance and relativity. A particularly important aspect of the action approach will be that it shows how one can introduce modifications to our theory of gravity in a way that insures consistency with general covariance and relativity. One then wonders — which, if any, of such modifications is “right”? Ultimately that is a question that must be addressed by experiment and observation.

## 12.6 A word on units

Moving forward, we will use Newton's gravitational constant  $G$  to help keep track of gravitational terms. However, it is very common to set both  $G$  and  $c$  to 1 in many calculations<sup>3</sup>. When we set  $G$  and  $c$  to 1, we find that mass, length, and time are all measured in the same<sup>4</sup> units.

A useful way to keep terms straight in your head is to note that factors of  $G$  and  $c$  can be used to convert between different kinds of units. For example, the combination  $G/c^2$  converts “normal” mass units to a length. A useful conversion factor in many astrophysical applications is

$$\frac{GM_{\odot}}{c^2} = 1.476 \text{ km} , \quad (12.40)$$

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<sup>3</sup>This in fact is the default in much of my research, so please watch in case I drop factors of  $G$  — it's so ingrained that I tend to forget once I've got some momentum at the blackboard.

<sup>4</sup>This contrasts with the system often used in quantum field theory: setting  $\hbar$  and  $c$  to 1, we find that the dimensions of mass (and energy) are the *inverse* of the dimensions of length and time.

where  $M_{\odot} \simeq 1.989 \times 10^{30}$  kg is the mass of the Sun. Another factor of  $c$  in the denominator converts masses to times:

$$\frac{GM_{\odot}}{c^3} = 4.924 \times 10^{-6} \text{ sec} . \quad (12.41)$$

When working in a problem dominated by the gravitational dynamics of a body with mass  $M$ , one often finds that important lengths in the problem scale with  $GM$ , or  $GM/c^2$ ; important times in the problem scale with  $GM$ , or  $GM/c^3$ . Frequencies associated with gravitational dynamics tend to scale as  $1/M$ , or  $c^3/GM$  (a factor that comes up a lot when thinking about gravitational-wave sources, for instance).

The factor  $G/c^4$  converts energy to length. This is an important one because the stress-energy tensor's components have the dimensions of energy density, or energy per length cubed. This means that  $GT_{\mu\nu}/c^4$  has dimensions  $(\text{length})^{-2}$ , which is exactly the dimensions we expect for *curvature*. The factor  $\kappa$  we introduced when developing the Einstein field equation is more completely written

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} . \quad (12.42)$$

Notice that, in “normal” units, this factor is very small:  $G/c^4 = 8.26 \times 10^{-45}$  m/J — it takes a *lot* of energy density to generate spacetime curvature. This is another way of expressing the fact that gravity is the weakest of the fundamental forces of nature. But, because mass and energy only have one sign, gravity adds up and can have a significant effect.