MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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LECTURE 13
ANOTHER APPROACH TO A THEORY OF GRAVITY

13.1 Field theories from Lagrangians

In the previous lecture, by assembling the various ingredients and tools we developed over the previous lectures and by insisting that we recover Newtonian gravity in an appropriate limit, we developed a field equation that relates the geometry of spacetime to the energy and momentum content of spacetime. This Einstein field equation is the core of Einstein's general theory of relativity; its presentation[1] is often taken to be the starting point of this subject. With this field equation in hand, we are ready to start developing solutions which describe gravity according to the principles we have been developing all semester, and to begin exploring how relativistic gravity differs from the Newtonian description.

Before doing so, it is useful to confront the fact that the derivation we used — though historically quite similar to how general relativity was first presented to the world — is somewhat dissatisfying. As we noted in the previous lecture, one could imagine other combinations of curvature tensors which reproduce the Newtonian limit and which satisfy the various principles which underlie relativistic physics. How do we choose?

As physicists, measurement and observation are the ultimate arbiters of how we assemble things. However, as *theoretical* physicists, we seek ways of organizing our model building to guide us toward plausible ways of understanding how spacetime *might* behave, subject to the various ideas or constraints we wish to put on the theory. The track we will follow in this lecture is one that Einstein's mathematician colleague David Hilbert originally developed, and published at nearly the same time in 1915 as Einstein published the field equations (though he always credited Einstein for the essential ideas that went into this analysis; they were in close contact during the years that general relativity was formulated). This approach is based on developing a Lagrangian density which describes gravitation.

Such an approach is now standard in much of field theory. The basic idea is to imagine that there is some function $\hat{\mathcal{L}}$ which characterizes the fields you wish to study, and that an action can be found by integrating this over all of spacetime:

$$S = \int d^4x\, \hat{\mathcal{L}}\ . \tag{13.1}$$

Since the action must be a Lorentz scalar, is not uncommon to write this

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}\ , \tag{13.2}$$

including the factor $\sqrt{-g}$ needed to make a proper volume element in a spacetime $g_{\mu\nu}$. We require that this action be stationary with respect to variations in the fields, so schematically we require

$$\delta S = \int d^4x \left[\frac{\partial \hat{\mathcal{L}}}{\partial(\text{fields})}\right] \delta(\text{fields}) = 0\ . \tag{13.3}$$

[1] A. Einstein, "Die Feldgleichungen der Gravitation," Sitzungsberichte der Preussischen Akademie der Wissenschaften, 844 (1915).

Enforcing

$$\frac{\partial \hat{\mathcal{L}}}{\partial(\text{fields})} = 0 \tag{13.4}$$

then yields Euler-Lagrange equations that describe the various fields that go into our theory.

To remind ourselves of how this works, let's imagine that $\mathcal{L}$ depends on scalar fields Φ^a and their derivatives $\partial_\mu \Phi^a$ in flat spacetime $\eta_{\mu\nu}$ (so that $\sqrt{-g} = 1$, and there is no distinction between $\mathcal{L}$ and $\hat{\mathcal{L}}$). Note that the superscript "a" just labels different fields, and is not intended to be a spacetime index. We put $\mathcal{L} = \mathcal{L}(\Phi^a, \partial_\mu \Phi^a)$, and in our variation we will treat the field and its derivatives as separate degrees of freedom.

We put $\Phi^a \to \Phi^a + \delta\Phi^a$, $\partial_\mu \Phi^a \to \partial_\mu \Phi^a + \partial_\mu(\delta\Phi^a)$, and so our variation in S becomes

$$\begin{aligned} \delta S &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^a} \delta\Phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \partial_\mu(\delta\Phi^a) \right] \\ &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) \right] \delta\Phi^a \ . \end{aligned} \tag{13.5}$$

On that last line, we have integrated by parts and assumed that there are no boundary terms, or that any such terms can be discarded. If the integral is over all of spacetime and we are working in the flat manifold of special relativity, then this is reasonable. In other circumstances, there may be subtle details that make this rather tricky; we will revisit this issue further below.

Requiring $\delta S = 0$, this analysis yields Euler-Lagrange equations for the field:

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) = 0 \ . \tag{13.6}$$

Further progress requires that we use some Lagrangian density for our fields (often just called a "field Lagrangian" or "Lagrangian"). A common example is the massive scalar field: focusing on a single field so we can drop the label a, we write down a Lagrangian with a kinetic term and a potential term as follows:

$$\mathcal{L} = -\frac{1}{2}\eta^{\mu\nu} \left(\partial_\mu \Phi\right) \left(\partial_\nu \Phi\right) - \frac{1}{2} m^2 \Phi^2 \ . \tag{13.7}$$

The derivatives which go into the Euler-Lagrange equation are

$$\frac{\partial \mathcal{L}}{\partial \Phi} = -m^2 \Phi \ , \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = -\eta^{\mu\nu} \partial_\nu \Phi \ . \tag{13.8}$$

Plugging these in, the Euler-Lagrange equation yields

$$\Box \Phi - m^2 \Phi = 0 \ , \tag{13.9}$$

where $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$ is the flat spacetime wave operator. This is simply the Klein-Gordon equation for a massive scalar field.

13.2 Constructing and varying an action for gravity: Version 1

We would like to develop a similar Lagrangian that is appropriate for a relativistic theory of gravity. Two main points guide our thinking:

- The action must be a scalar, so the Lagrangian must also be a scalar.
- To make sure that the equivalence principle is respected, this Lagrangian must be built out of curvature scalars so that it is not something that can be eliminated at any point simply by changing representation.

The suggests that the simplest possible action we can write down is

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R = \frac{1}{16\pi G}\int d^4x\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta}\ . \tag{13.10}$$

The factor of $1/G$ (which is c^4/G in "normal" units) insures that this final result has dimensions of action. The numerical factor, $1/16\pi$, is inserted more or less magically for later convenience. (We should really treat it as some undetermined number, and determine it later; I am skipping over these details because we've already done a version of this dance once.)

We will vary this action with respect to the metric. It turns out to be most convenient to do so using the inverse metric, $g^{\alpha\beta}$; we thus seek to compute

$$\delta S = \frac{1}{16\pi G}\int d^4x\frac{\partial}{\partial g^{\alpha\beta}}\left[\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta}\right]\delta g^{\alpha\beta}\ . \tag{13.11}$$

Let us examine variations in that integrand:

$$\delta\left(\sqrt{-g}g^{\alpha\beta}R_{\alpha\beta}\right) = \left(\delta\sqrt{-g}\right)R + \sqrt{-g}\,\delta g^{\alpha\beta}\,R_{\alpha\beta} + \sqrt{-g}g^{\alpha\beta}\,\delta R_{\alpha\beta}\ . \tag{13.12}$$

The first term in (13.12) can be found using some of the tricks associated with the determinant of the metric we discussed in an earlier lecture; we find that

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta} \tag{13.13}$$

and so

$$\left(\delta\sqrt{-g}\right)R = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\,R\,\delta g^{\alpha\beta}\ . \tag{13.14}$$

The second term of (13.12) is already in a form that is proportional to $\delta g^{\alpha\beta}$, and so requires no further massaging.

The third term, however, requires some work. To find the variation in the Ricci tensor, begin by looking at variations in Riemann:

$$\delta R^\mu{}_{\alpha\nu\beta} = \partial_\nu\delta\Gamma^\mu{}_{\beta\alpha} - \partial_\beta\delta\Gamma^\mu{}_{\nu\alpha} + \delta\Gamma^\mu{}_{\nu\gamma}\Gamma^\gamma{}_{\beta\alpha} + \Gamma^\mu{}_{\nu\gamma}\delta\Gamma^\gamma{}_{\beta\alpha} - \delta\Gamma^\mu{}_{\beta\gamma}\Gamma^\gamma{}_{\nu\alpha} - \Gamma^\mu{}_{\beta\gamma}\delta\Gamma^\gamma{}_{\nu\alpha}\ . \tag{13.15}$$

Although Christoffel symbols are not tensors, *differences* in Christoffel symbols are tensors: the pieces of the transformation rule that break the pattern needed to define a quantity as a tensor cancel out when we look at such a quantity. This means that $\delta\Gamma^\mu{}_{\beta\alpha}$ is tensorial, and we can sensibly define its covariant derivative:

$$\nabla_\nu\delta\Gamma^\mu{}_{\beta\alpha} = \partial_\nu\delta\Gamma^\mu{}_{\beta\alpha} + \delta\Gamma^\gamma{}_{\beta\alpha}\Gamma^\mu{}_{\nu\gamma} - \delta\Gamma^\mu{}_{\gamma\alpha}\Gamma^\gamma{}_{\nu\beta} - \delta\Gamma^\mu{}_{\beta\gamma}\Gamma^\gamma{}_{\nu\alpha}\ . \tag{13.16}$$

Using this expression, we can rewrite (13.15) using covariant derivatives rather than partial derivatives. Doing so, a miracle occurs: all the terms of the form $\delta\Gamma\,\Gamma$ cancel out, and we are left with

$$\delta R^\mu{}_{\alpha\nu\beta} = \nabla_\nu\delta\Gamma^\mu{}_{\beta\alpha} - \nabla_\beta\delta\Gamma^\mu{}_{\nu\alpha}\ . \tag{13.17}$$

Contracting on indices 1 and 3, we get the variation in Ricci:

$$\delta R_{\alpha\beta} = \nabla_\mu\delta\Gamma^\mu{}_{\beta\alpha} - \nabla_\beta\delta\Gamma^\mu{}_{\mu\alpha}\ . \tag{13.18}$$

We are not done: we still need to examine the variation in the Christoffel symbol. With some effort, one can show that

$$\delta\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}\left[\nabla_\gamma\left(g_{\alpha\nu}g_{\beta\lambda}g^{\mu\gamma}\delta g^{\nu\lambda}\right) - \nabla_\alpha\left(g_{\beta\gamma}\delta g^{\mu\gamma}\right) - \nabla_\beta\left(g_{\alpha\gamma}\delta g^{\mu\gamma}\right)\right]\ . \tag{13.19}$$

(Note that, although $\nabla_\alpha g_{\mu\nu} = 0$, we cannot assume $\nabla_\alpha \delta g^{\mu\nu} = 0$.) Using this, one can finally assemble

$$g^{\alpha\beta}\delta R_{\alpha\beta} = \nabla_\alpha \nabla_\beta \left(g^{\alpha\beta} g_{\mu\nu} \delta g^{\mu\nu} - \delta g^{\alpha\beta}\right) \equiv \nabla_\alpha v^\alpha \ . \tag{13.20}$$

This final form recognizes that the expression we at last landed on can be thought of as the covariant divergence of a vector field v^α whose form can be read out of Eq. (13.20).

Putting these three terms together, the variational principle we seek to enforce takes the form

$$\begin{aligned}\delta S &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[\left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R\right)\delta g^{\alpha\beta} + \nabla_\alpha v^\alpha\right] \\ &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[G_{\alpha\beta}\delta g^{\alpha\beta} + \nabla_\alpha v^\alpha\right] \ . \end{aligned} \tag{13.21}$$

The first term inside the square brackets looks pretty much just like what we want. The second term does not. Because this term is in the form of a divergence, one can imagine invoking the divergence theorem and converting it into a boundary term. We presumably could then discard this term by requiring the variation to vanish on the boundary.

For now, this argument is fine; however, it is at best somewhat glib. We can set variations in the metric to zero at the boundary, but we cannot set both variations of the metric *and* its derivative to zero — variations in the metric's derivative are not part of our variational scheme. To handle this term rigorously, we need to examine boundary regions in our analysis very carefully. This requires that we carefully define how to compute the contribution to curvature that arises for a 3-dimensional "slice" of spacetime carved out of our 4-dimensional manifold. (Or, we need to modify our variational scheme; such a modification is introduced later in these lecture notes.)

Students who wish to examine how to handle the boundary in (much) more detail are invited to carefully examine the discussion in Appendix E of Wald's textbook. For our purposes, we take the glib assertion that we can discard the $\nabla_\alpha v^\alpha$ term to be good enough[2] for the purposes of 8.962. Enforcing $\delta S = 0$, Eq. (13.21), then yields $G_{\alpha\beta} = 0$ — the Einstein field equation with no source.

13.3 Putting in a source

More generally, we can write our action for everything — matter, fields, spacetime — in the form

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{16\pi G} + \mathcal{L}_{M/F}\right) \ , \tag{13.22}$$

where $\mathcal{L}_{M/F}$ is a Lagrangian for all matter or fields that live in spacetime. We then define the stress-energy tensor to be

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L}_{M/F})}{\partial g^{\alpha\beta}} \ . \tag{13.23}$$

The somewhat funny factors of -2 and $\sqrt{-g}$ are included to account for the terms which arise from varying the term which gives us a properly tensorial volume element. At this stage, we recognize the factor $1/16\pi$ that we included earlier as what was necessary in order for our field equation to have the correct Newtonian limit.

Carroll works out an example of how this variational principle duplicates the stress-energy tensor for a scalar field; see discussion from Eq. (4.76) until the end of Sec. 4.3 in his textbook. We work through another illustrative example here: consider

$$\mathcal{L}_{M/F} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \ . \tag{13.24}$$

[2] Vladimir Braginsky (`https://en.wikipedia.org/wiki/Vladimir_Braginsky`) used to attend meetings of the group in which I was a grad student, and once memorably advised me in a wonderfully thick Russian accent that "Perfect is enemy of Good Enough."

This describes the electromagnetic field. Put it inside the action integral, and write it as

$$S = \int d^4x\sqrt{-g}\, g^{\alpha\mu} g^{\beta\nu}\, F_{\alpha\beta}F_{\mu\nu}\ . \tag{13.25}$$

We now vary the spacetime metric, but leave the term $F_{\alpha\beta}F_{\mu\nu}$ unvaried. The variation we need is

$$\delta\left[\sqrt{-g}\, g^{\alpha\mu} g^{\beta\nu}\right] = -\frac{1}{2}\sqrt{-g}(g_{\sigma\rho}\delta g^{\sigma\rho})g^{\alpha\mu}g^{\beta\nu} + \sqrt{-g}g^{\alpha\mu}\delta g^{\beta\nu} + \sqrt{-g}g^{\beta\nu}\delta g^{\alpha\mu}\ . \tag{13.26}$$

Applying this variation to the action integral, we have

$$\delta S = -\frac{1}{4}\int d^4x\sqrt{-g}\left[F_{\alpha\beta}F^{\alpha}{}_{\nu}\,\delta g^{\beta\nu} + F_{\alpha\beta}F_{\mu}{}^{\beta}\,\delta g^{\alpha\mu} - \frac{1}{2}F^{\mu\nu}F_{\mu\nu}g_{\sigma\rho}\delta g^{\sigma\rho}\right]\ . \tag{13.27}$$

We cycle dummy indices in each term in order to pull out a common factor of $\delta g^{\alpha\beta}$ and to facilitate combining some of the terms under the integrand: in the first term in square brackets under the integrand of (13.27), we take $\alpha \to \mu$, $\beta \to \alpha$, $\nu \to \beta$; in the second term, we take $\mu \to \beta$, $\beta \to \mu$; and in the third, we take $\sigma \to \alpha$, $\rho \to \beta$. This yields

$$\delta S = -\frac{1}{4}\int d^4x\sqrt{-g}\left[F_{\mu\alpha}F^{\mu}{}_{\beta} + F_{\alpha\mu}F_{\beta}{}^{\mu} - \frac{1}{2}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right]\delta g^{\alpha\beta}\ . \tag{13.28}$$

Finally using the antisymmetry of the electromagnetic field tensor, we write this

$$\delta S = -\frac{1}{2}\int d^4x\sqrt{-g}\left[F_{\mu\alpha}F^{\mu}{}_{\beta} - \frac{1}{4}g_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right]\delta g^{\alpha\beta} \equiv -\frac{1}{2}\int d^4x\sqrt{-g}\left[T^{\rm EM}_{\alpha\beta}\right]\delta g^{\alpha\beta}\ . \tag{13.29}$$

On the final line, we used the definition (13.23) to recognize that the quantity in square brackets in Eq. (13.29) is the electromagnetic stress-energy tensor.

13.4 Constructing and varying an action for gravity: Version 2

One of the reasons that the boundary term got us into difficulty in the variation we studied above is that we imagine the "field" that we vary to be the metric of spacetime. Doing so, we can impose a requirement that it not vary on the boundary of some region. This requirement does not constrain derivatives of the metric.

What if we treat the metric and its derivatives as independent degrees of freedom, akin to the variation of the scalar field that we examined at the beginning of these notes? More concretely, imagine that S depends *separately* on the metric and on the connection, and that we *separately* vary these two quantities. We write

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\, R(\Gamma) = \frac{1}{16\pi G}\int d^4x\sqrt{-g}g^{\alpha\beta}\, R_{\alpha\beta}(\Gamma)\ . \tag{13.30}$$

The notation $R(\Gamma)$, $R_{\alpha\beta}(\Gamma)$ indicates somewhat schematically that the Ricci tensor and Ricci scalar are functions of the connection. We now imagine that the tensor $R_{\alpha\beta}$ *does not* change as we vary the metric. We also do not assume that the metric and the covariant derivative are necessarily compatible, i.e. we do not assume that $\nabla_\alpha g_{\mu\nu} = 0$. That would assume a particular value for the connection, and we wish to leave the connection unspecified until the variational principle guides us to its value.

With this idea in mind, let us revisit the variation. First, vary the action with respect to the metric. Begin by writing

$$\delta_g S = \int d^4x\left[\delta_g(\sqrt{-g})R + \sqrt{-g}R_{\alpha\beta}\,\delta g^{\alpha\beta}\right]\ . \tag{13.31}$$

We expand that first variation using

$$\delta_g\left(\sqrt{-g}\right) = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta} \ , \tag{13.32}$$

as we worked out before. Nothing else varies here, since the Ricci tensor depends on the connection Γ rather than the metric g. With this, the variation of the action with the metric becomes

$$\begin{aligned}\delta_g S &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\left[R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right]\delta g^{\alpha\beta} \\ &= \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,G_{\alpha\beta}\,\delta g^{\alpha\beta} \ .\end{aligned} \tag{13.33}$$

Requiring that this vanish for any variation in the metric yields the vacuum field equation, $G_{\alpha\beta} = 0$. Including other matter or fields and following the procedure described in the previous section to produce the stress energy tensor yields the complete field equation, $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$.

Now consider varying the connection. Begin with the fact that, as discussed above, the variation in the Ricci curvature may be written

$$\delta R_{\alpha\beta} = \nabla_\mu \delta\Gamma^\mu{}_{\beta\alpha} - \nabla_\beta \delta\Gamma^\mu{}_{\mu\alpha} \ . \tag{13.34}$$

In what follows, let us explicitly indicate that we are integrating over some 4-volume V^4. We will require that the variation in the connection vanish on the boundary of the 4-volume V^4, i.e. that $\delta\Gamma^\mu{}_{\alpha\beta} \to 0$ on ∂V^4. The variation in the action upon varying the connection is given by

$$\delta_\Gamma S = \frac{1}{16\pi G}\int_{V^4} d^4x\left[\nabla_\mu\delta\Gamma^\mu{}_{\beta\alpha} - \nabla_\beta\delta\Gamma^\mu{}_{\mu\alpha}\right]\sqrt{-g}g^{\alpha\beta} \ . \tag{13.35}$$

Because variations in the connection are tensorial, we can expand the covariant derivative in the usual way. Let us do so, writing

$$\nabla_\mu\delta\Gamma^\mu{}_{\beta\alpha} = \partial_\mu\delta\Gamma^\mu{}_{\beta\alpha} + \text{terms of the form } \delta\Gamma\,\Gamma \ . \tag{13.36}$$

A similar form describes the other covariant derivative. Although the terms of the form $\delta\Gamma\,\Gamma$ are only indicated here schematically, it is important to get them right for the analysis which follows. I won't fill in all the algebra, which though straightforward is somewhat tedious, but just outline the procedure to be followed.

The terms which involve partial derivatives can now be integrated by parts. Analysis of the one we have written out proceeds as follows:

$$\int_{V^4} d^4x\,\partial_\mu\delta\Gamma^\mu{}_{\beta\alpha}\,g^{\alpha\beta}\sqrt{-g} = \int_{\partial V^4}\delta\Gamma^\mu{}_{\beta\alpha}\,g^{\alpha\beta}\,d\Sigma_\mu - \int_{V^4} d^4x\,\delta\Gamma^\mu{}_{\beta\alpha}\,\partial_\mu\left(g^{\alpha\beta}\sqrt{-g}\right) \ . \tag{13.37}$$

Note that in the first integral, we've absorbed a factor of $\sqrt{-g}$ into the definition of the "surface" element $d\Sigma_\mu$. The boundary term contributes nothing by virtue of the fact that the variation in the connection vanishes on ∂V^4; what remains is an integral of the variation against the partial derivative of the (inverse) metric times the determinant factor.

Doing a similar exercise for the other covariant derivative, putting all terms together, gathering together all the terms of the form $\delta\Gamma\,\Gamma$, and finally relabeling dummy indices, an apparent miracle occurs: *the whole thing can be written as variations in the connection linked to* **covariant** *derivatives of* $g^{\alpha\beta}\sqrt{-g}$. In particular, what we find is

$$\begin{aligned}\delta_\Gamma S &= \frac{1}{16\pi G}\int_{V^4} d^4x\left[\delta\Gamma^\mu{}_{\mu\alpha}\nabla_\beta\left(g^{\alpha\beta}\sqrt{-g}\right) - \delta\Gamma^\mu{}_{\beta\alpha}\nabla_\mu\left(g^{\alpha\beta}\sqrt{-g}\right)\right] \\ &= \frac{1}{16\pi G}\int_{V^4} d^4x\,\delta\Gamma^\mu{}_{\nu\alpha}\left[\delta^\nu{}_\mu\nabla_\beta\left(g^{\alpha\beta}\sqrt{-g}\right) - \delta^\nu{}_\beta\nabla_\mu\left(g^{\alpha\beta}\sqrt{-g}\right)\right] \ .\end{aligned} \tag{13.38}$$

On the second line, we've used the Kronecker delta and relabeled dummy indices to pull out a common factor of the variation in the connection.

This variation is stationary if the term in brackets vanishes, which is the case if

$$\nabla_\mu \left(g^{\alpha\beta}\sqrt{-g} \right) = 0 \; . \tag{13.39}$$

This condition is in fact met if our connection coefficients are equal to the Christoffel synbols. This demonstrates that this *Palatini variation* eliminates the problematic boundary term that arises when we vary only the metric, and yields the Christoffel connection as a choice compatible with the variation[3].

13.5 Advantage of the variational approach

Both of the variational frameworks we discussed yield the Einstein field equations, as does the "historical" approach discussed in the previous lecture. So what do we gain by going through all this additional analysis?

As emphasized earlier, the properties of any fundamental interaction such as gravity are ultimately ones that must be determined empirically; but, as theorists we seek a framework that allows us to "catalog" what properties may exist given particular assumptions about the properties this interaction must satisfy. The Einstein-Hilbert Lagrangian, $\mathcal{L} = R/(16\pi G)$, is the *simplest* one which is fully covariant and assembled only from curvature tensors. Since varying this action yields the Einstein field equation, this tells us that the theory of gravity arising from this field equation — general relativity — is in a very fundamental sense the simplest possible theory of gravity compatible with covariance and the equivalence principle.

One could imagine other possibilities, though. For example, what happens if we choose

$$\mathcal{L} = \frac{1}{16\pi G}\left(R - \frac{\alpha}{R} \right) \tag{13.40}$$

as our gravitational Lagrangian? Heuristically, this choice indicates that we expect things to look "just like" general relativity when $R \gg \sqrt{\alpha}$, but have very different behavior when $R \lesssim \sqrt{\alpha}$. This Lagrangian has introduced a particular curvature scale at which we expect gravitational dynamics to change character. One can show[4] that the field equations produced by this Lagrangian are

$$G_{\mu\nu} + \frac{\alpha}{R^2}\left[R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \right] + \alpha \left[g_{\mu\nu}\nabla_\sigma \nabla^\sigma - \nabla_{(\mu}\nabla_{\nu)} \right] R^{-2} = 8\pi G T_{\mu\nu} \; . \tag{13.41}$$

This describes a theory of gravity for which the dynamics are identical to those of general relativity at "large curvature," but things are quite different for "small curvature." This Lagrangian was proposed as a way to explain the accelerated expansion of the universe. It was eventually falsified for being incompatible with measurements of the precession of orbits in our solar system — a textbook example of how to modify a theory in a way that is compatible with important physical principles, then empirically testing whether those modifications fit what we measure or not.

One can also imagine a Lagrangian that involves higher-order curvature terms. The most general such form is

$$\mathcal{L} = \frac{1}{16\pi G}\left(R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\mu\sigma\nu\rho}R^{\mu\sigma\nu\rho} \right) \; . \tag{13.42}$$

There is some theoretical prejudice that $\alpha_{1,2,3} \sim \ell_{\rm P}^2 \sim \hbar$, where $\ell_{\rm P}$ is the *Planck length*. A fair amount of literature and work these days examines the possibility of *Lovelock gravity*, a proposal that these curvature terms enter the Lagrangian in the form

$$\mathcal{L} = \frac{1}{16\pi G}\left[R + \alpha\left(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\sigma\nu\rho}R^{\mu\sigma\nu\rho} \right] \right) \; . \tag{13.43}$$

[3]Interestingly, there are other choices that could be made to solve $\nabla_\mu(g^{\alpha\beta}\sqrt{-g}) = 0$, though discussing them goes well beyond our needs. See J. T. Wheeler, arXiv:2201.00938 for further discussion.

[4]See S. M. Carroll et al, Phys. Rev. D **70**, 043528 (2004).

Such a choice arises because the term after α is used to compute the Euler-Poincaré characteristic of curved manifolds, and is known as the Gauss-Bonnet term.

One can also imagine that gravity is not purely a consequence of spacetime geometry, but that some field (often taken to be a scalar) exists which couples to curvature. The gravity Lagrangian is then given by

$$\mathcal{L}_g = \frac{1}{16\pi G} f(\Phi) R \,, \tag{13.44}$$

which is sometimes usefully thought of as the coupling constant G become a quantity that depends on the scalar field:

$$\mathcal{L}_g = \frac{1}{16\pi G(\Phi)} R \,. \tag{13.45}$$

This must be supplemented by a Lagrangian describing the dynamics of the scalar field:

$$\mathcal{L}_\Phi = h(\Phi) \nabla^\alpha \nabla_\alpha \Phi - V(\Phi) \,, \tag{13.46}$$

with the principles of your *scalar-tensor* theory providing the functions $f(\Phi)$ and $h(\Phi)$, as well as the potential $V(\Phi)$. Note that such theories in general violate the equivalence principle — different bodies may fall at different rates depending on how they couple to Φ. Further discussion can be found in Sec. 4.8 of Carroll.

From our standpoint, the take-away message is that this framework gives us a way of developing a description of gravity which goes beyond general relativity, but does so in a methodical way that builds into the analysis from the start whatever principles you wish your theory of gravity to incorporate. For the purpose of 8.962, we will henceforth focus solely on the unadorned theory of general relativity; but the tools exist to examine how things differ if we deviate from this surprisingly simple starting point.