MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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LECTURE 14
SOLVING THE EINSTEIN FIELD EQUATIONS: WEAK FIELD SOLUTIONS

14.1 Solving the Einstein field equation: General considerations

For the remainder of 8.962, we will focus on solving the Einstein field equation $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ and studying the properties of its solutions (including the cosmological constant in our analysis for one particular class of solutions).

When this equation is regarded as a statement about the geometry of spacetime and how it relates to the flow and density of energy and momentum in this spacetime, terms like "simple" and "beautiful" tend to be used. Indeed, as we saw in Lecture 13, Einstein's general relativity is in a very real sense the simplest possible theory of gravity that respects general covariance and the principle of equivalence, arising from the simplest Lagrangian that can be constructed from curvature. The result is a straightforward equivalence between a divergence-free tensor that describes spacetime curvature and the flow of energy and momentum through spacetime.

On the other hand, if we regard the field equations as differential equations that specify the spacetime metric — which is often what we want for the purpose of physical analysis — then they are a rather horribly complicated mess. Consider the situation for a perfect fluid. On the left-hand side, we have the Einstein tensor, which we can regard as a differential operator acting on the metric of spacetime:

$$G_{\mu\nu} = \mathcal{D}^2_{\mu\nu}\left[g_{\alpha\beta}\right] \ . \tag{14.1}$$

In this schematic form, $\mathcal{D}^2_{\mu\nu}$ is a 2nd-order, nonlinear partial differential operator. On the right, we have

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + P g_{\mu\nu} \ . \tag{14.2}$$

The spacetime metric thus appears explicitly on the right-hand side, as well as under the operator on the left-hand side. It is also present *implicitly* on the right-hand side via the requirement that the four velocity be properly normalized: $u^\mu u^\nu g_{\mu\nu} = -1$.

So as a differential equation for the metric, the Einstein field equation sets us up for what appears likely to be a rather challenging analysis. To make progress, we will implement three approaches to solving this equation:

1. *"Weak" gravity.* Here we imagine that spacetime is not very different (a statement that must be made quantitative) from that of special relativity, at least in some coordinate systems, and we will linearize the field equations in that deviation from $\eta_{\mu\nu}$. Such an analysis can be regarded as the first order in a perturbative expansion. We will discuss in schematic terms how one can iterate this expansion to higher order in the last week of this course.

2. *Impose a symmetry.* By imposing a symmetry, the complexity of the operator $\mathcal{D}^2_{\mu\nu}$ and of any source terms is greatly reduced. The coupled nonlinear nastiness that remains is one that can be handled without too much trouble; this is how some of the most important solutions in general relativity have been constructed. By adding perturbations around these symmetric but exact solutions, we can do a tremendous amount of useful analysis. (Indeed, a rather large chunk of your lecturer's career is based on studying perturbations around a particular exact solution.)

3. *Just deal with it: Code up the equations and attack.* Numerical analysis can be a very powerful tool, provided we understand its limitations and can assess whether our solutions are converging to a description of the correct physics. Especially since about 2005, numerical methods for solving the Einstein field equations have been particularly productive; but it took decades of foundational work to understand how to cast the equations into a form that was amenable to this method of building solutions. We will discuss work in this vein in the last week of this course.

We begin by studying the linearized field equations.

14.2 The "weak-field" limit: Linearized gravity

The weak-field limit is defined by the idea that spacetime is *nearly* flat. We choose coordinates such that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \;, \tag{14.3}$$

and in which the components describing the deviation from flatness are small enough that we can linearize with negligible error: $||h_{\mu\nu}|| \ll 1$, so terms $\mathcal{O}(h^2)$ can be neglected. Whether this linearization is accurate or not is ultimately a matter of measurement precision. It is conceivable that measurements which agree with the predictions of linearized theory will cease to agree if measurement techniques are significantly improved. In this part of 8.962, we take the ansatz that the linear approximation is acceptable, and develop this limit of general relativity in detail.

Coordinates in which one can write the spacetime metric in the form (14.3) are known as "nearly Lorentz coordinates." One can think of them as coordinates that hold in a convenient freely falling frame and are then extended to cover the entire manifold of (nearly flat) spacetime. Note that we have borrowed some of the ideas and results from this analysis in earlier sections of this course, particularly in making sure that our field equation included a Newtonian limit. Our goal now is to put this framework from which we have been borrowing onto solid ground.

Because the spacetime metric is a tensor, it of course is subject to the general rule for changing representation:

$$g_{\bar\mu\bar\nu} = \frac{\partial x^\alpha}{\partial x^{\bar\mu}}\frac{\partial x^\beta}{\partial x^{\bar\nu}} g_{\alpha\beta} \;. \tag{14.4}$$

When the spacetime is flat, we picked out a special category of representation changes, the Lorentz transformations, as playing an important role. One thing that makes the Lorentz transformations special is that the metric has the same representation in all Lorentz frames:

$$\eta_{\bar\mu\bar\nu} = \Lambda^\alpha{}_{\bar\mu}\Lambda^\beta{}_{\bar\nu}\eta_{\alpha\beta} \;, \tag{14.5}$$

and both $\eta_{\bar\mu\bar\nu}$ and $\eta_{\alpha\beta}$ are represented by the matrix $\mathrm{diag}(-1, 1, 1, 1)$.

The Lorentz transformation really only has physical meaning in special relativity — in general relativity, it is not generally applicable except within the confines of a local Lorentz frame. However, in the case of nearly flat spacetimes, we can extend this concept in a useful way. Begin by applying the Lorentz transformation to a nearly flat spacetime metric:

$$\begin{aligned} g_{\bar\mu\bar\nu} &= \Lambda^\alpha{}_{\bar\mu}\Lambda^\beta{}_{\bar\nu}\left(\eta_{\alpha\beta} + h_{\alpha\beta}\right) \\ &= \eta_{\bar\mu\bar\nu} + \Lambda^\alpha{}_{\bar\mu}\Lambda^\beta{}_{\bar\nu} h_{\alpha\beta} \\ &\equiv \eta_{\bar\mu\bar\nu} + h_{\bar\mu\bar\nu} \;. \end{aligned} \tag{14.6}$$

The background flat spacetime is unchanged by the Lorentz transformation, but the "perturbation" $h_{\alpha\beta}$ transforms just like any tensor field in special relativity. In this circumstance, we call this transformation a *background* Lorentz transformation, since we are taking advantage of the fact that the flat background to our perturbation respects Lorentz invariance.

The relationship $h_{\bar{\mu}\bar{\nu}} = \Lambda^{\alpha}{}_{\bar{\mu}}\Lambda^{\beta}{}_{\bar{\nu}}h_{\alpha\beta}$ tells us that a useful fiction is to regard $h_{\alpha\beta}$ as simply a tensor field that lives in flat spacetime. This is a fiction because in fact the spacetime is curved, but it is a useful one for us because it gives us a calculational "narrative" that is very similar to how we describe many field theories in special relativity.

Another operation that we need to work out is the form of the inverse metric in linearized theory. We begin by raising indices:

$$\begin{aligned} h^{\alpha\beta} &= g^{\alpha\mu}g^{\beta\nu}h_{\mu\nu} \\ &= \eta^{\alpha\mu}\eta^{\beta\nu}h_{\mu\nu} + \mathcal{O}(h^2) \, . \end{aligned} \tag{14.7}$$

We always find that indices related to spacetime tensors are raised and lowered using the background metric to linear order, which is consistent with the idea that we can treat $h_{\mu\nu}$ as a tensor field in a flat background. With this in hand, let us calculate the inverse metric:

$$\begin{aligned} \delta^{\alpha}{}_{\gamma} &= g^{\alpha\beta}g_{\beta\gamma} \\ &= \left(\eta^{\alpha\beta} + m^{\alpha\beta}\right)\left(\eta_{\beta\gamma} + h_{\beta\gamma}\right) \, . \end{aligned} \tag{14.8}$$

The first line here is just the definition of the metric inverse $g^{\alpha\beta}$. On the second line, we have introduced a tensor $m^{\alpha\beta}$ whose elements are of the same order as the elements of $h_{\alpha\beta}$. We leave it undetermined in order to see what value it must have in order to provide the correct inverse metric. Expanding, we have

$$\delta^{\alpha}{}_{\gamma} + h^{\alpha}{}_{\gamma} + m^{\alpha}{}_{\gamma} + \mathcal{O}(h^2) = \delta^{\alpha}{}_{\gamma} \, , \tag{14.9}$$

which tells us that

$$h^{\alpha}{}_{\gamma} = -m^{\alpha}{}_{\gamma} \tag{14.10}$$

to linear order. Therefore,

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} \tag{14.11}$$

to linear order in the deviation from $\eta_{\alpha\beta}$. This is a form that we used earlier without careful justification in our analysis of the Newtonian limit, although this result should intuitively remind you of the functional expansion $1/(1+x) \simeq 1 - x$ for small x.

14.3 Gauge transformations in linearized gravity

Consider next making a coordinate shift, defined as

$$x^{\alpha'} = x^{\alpha} + \xi^{\alpha}(x^{\beta}) \, . \tag{14.12}$$

It must be acknowledged that this is not the best notation — this form abuses the index notation somewhat. Calling the new coordinate $(x')^{\alpha}$ rather than $x^{\alpha'}$ would be more consistent with the way we use indices in relativity. However, it is best to live with this inconsistency. Our goal here is not to introduce a covariant relationship, but simply to indicate how two coordinate systems are related to one another. In particular, one of the reasons that this somewhat irritatingly inconsistent notation is commonly used is that we wish to introduce the coordinate transformation matrix

$$L^{\alpha'}{}_{\beta} = \delta^{\alpha}{}_{\beta} + \partial_{\beta}\xi^{\alpha} \, . \tag{14.13}$$

We imagine that both the new and the old coordinates are nearly inertial, and require the matrix of partials to be very small: $||\partial_{\alpha}\xi^{\beta}|| \ll 1$. We call this an "infinitesimal" coordinate transformation. Using these properties, it is simple to show that the inverse transformation is given by

$$L^{\alpha}{}_{\beta'} = \delta^{\alpha}{}_{\beta} - \partial_{\beta}\xi^{\alpha} + \mathcal{O}[(\partial\xi)^2] \, . \tag{14.14}$$

Let us now examine how the metric changes under this coordinate transformation:

$$\begin{aligned}
g_{\mu'\nu'} &= L^\alpha{}_{\mu'} L^\beta{}_{\nu'} g_{\alpha\beta} \\
&= (\delta^\alpha{}_\mu - \partial_\mu \xi^\alpha)(\delta^\beta{}_\nu - \partial_\nu \xi^\beta)(\eta_{\alpha\beta} + h_{\alpha\beta}) \\
&= \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \mathcal{O}(h\,\partial\xi)\ .
\end{aligned} \tag{14.15}$$

The effect of this coordinate transformation is to change the perturbation around flat spacetime:

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu\ . \tag{14.16}$$

This should remind you of a gauge transformation in electrodynamics, in which one adjusts the potential $A_\mu \to A_\mu - \partial_\mu \Lambda$. We thus call this operation a gauge transformation in linearized gravity.

Just as the gauge transformation in electrodynamics changes potentials but leaves the field unchanged, the linearized gravity gauge transformation has no effect on curvature. It is simple to show that the Riemann tensor is given by

$$R_{\mu\alpha\nu\beta} = \frac{1}{2}\left(\partial_\alpha \partial_\nu h_{\mu\beta} + \partial_\mu \partial_\beta h_{\alpha\nu} - \partial_\alpha \partial_\beta h_{\mu\nu} - \partial_\mu \partial_\nu h_{\alpha\beta}\right) \tag{14.17}$$

in linearized theory; just expand the formula given in terms of Christoffel symbols, discarding the nonlinear terms and using the fact that we raise and lower indices with the flat spacetime metric (which commutes with partial derivatives since it is a constant). If one changes the metric according to (14.16), then Riemann is apparently shifted by

$$\begin{aligned}
\delta R_{\mu\alpha\nu\beta} &= \frac{1}{2}\Big(-\partial_\alpha \partial_\nu \partial_\mu \xi_\beta - \partial_\alpha \partial_\nu \partial_\beta \xi_\mu - \partial_\mu \partial_\beta \partial_\alpha \xi_\nu - \partial_\mu \partial_\beta \partial_\nu \xi_\alpha \\
&\qquad + \partial_\mu \partial_\nu \partial_\alpha \xi_\beta + \partial_\alpha \partial_\beta \partial_\nu \xi_\mu + \partial_\alpha \partial_\beta \partial_\mu \xi_\nu + \partial_\mu \partial_\nu \partial_\eta \xi_\alpha\Big)\ .
\end{aligned} \tag{14.18}$$

Because partial derivatives commute, one can see that $\delta R_{\mu\alpha\nu\beta} = 0$. We will take advantage of this property to adjust our gauge in such a way that curvature tensors, written as derivatives of the metric, take a particularly convenient form for our analysis.

14.4 The linear gravity field equation and its leading solution

Having already built Riemann in order to examine its behavior under gauge transformations, let's build the various contractions that we need for the field equation. We start with the Ricci tensor:

$$\begin{aligned}
R_{\alpha\beta} &= \eta^{\mu\nu} R_{\mu\alpha\nu\beta} \\
&= \frac{1}{2}\left(\partial_\alpha \partial^\mu h_{\mu\beta} + \partial_\beta \partial^\mu h_{\mu\alpha} - \partial_\alpha \partial_\beta h - \Box h_{\alpha\beta}\right)\ .
\end{aligned} \tag{14.19}$$

Here $\Box = \partial^\mu \partial_\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu$ is the wave operator in flat spacetime, and we have introduced $h = h^\mu{}_\mu = \eta^{\mu\nu} h_{\mu\nu}$, the trace of the metric perturbation.

The Ricci scalar follows simply from this:

$$R = \eta^{\alpha\beta} R_{\alpha\beta} = \partial^\alpha \partial^\mu h_{\mu\alpha} - \Box h\ . \tag{14.20}$$

Finally, the linearized Einstein tensor is given by

$$\begin{aligned}
G_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} R \\
&= \frac{1}{2}\left(\partial_\alpha \partial^\mu h_{\mu\beta} + \partial_\beta \partial^\mu h_{\mu\alpha} - \partial_\alpha \partial_\beta h - \Box h_{\alpha\beta} + \eta_{\alpha\beta} \Box h - \eta_{\alpha\beta} \partial^\mu \partial^\nu h_{\mu\nu}\right)\ .
\end{aligned} \tag{14.21}$$

The presence of the trace h makes this expression somewhat unwieldy. Recalling that $G_{\alpha\beta}$ is a trace-reversed version of the Ricci curvature, it is worth seeing what happens if we build it using a trace-reversed version of the metric perturbation. We define

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h \; ; \tag{14.22}$$

notice that $\eta^{\alpha\beta}\bar{h}_{\alpha\beta} = h - 2h = -h$. We then write $h_{\alpha\beta} = \bar{h}_{\alpha\beta} + \frac{1}{2}\eta_{\alpha\beta}h$, and substitute this into the Einstein tensor (14.21). The result is

$$\begin{aligned} G_{\alpha\beta} &= \frac{1}{2}\Big(\partial_\alpha\partial^\mu\bar{h}_{\mu\beta} + \frac{1}{2}\eta_{\mu\beta}\partial_\alpha\partial^\mu h + \partial_\beta\partial^\mu\bar{h}_{\mu\alpha} + \frac{1}{2}\eta_{\mu\alpha}\partial_\beta\partial^\mu h - \partial_\alpha\partial_\beta h - \Box\bar{h}_{\alpha\beta} \\ &\quad -\frac{1}{2}\eta_{\alpha\beta}\Box h + \eta_{\alpha\beta}\Box h - \eta_{\alpha\beta}\partial^\mu\partial^\nu\bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\mu\nu}\partial^\mu\partial^\nu h\Big) \; . \end{aligned} \tag{14.23}$$

Carefully examining this expression, we see that all the terms involving the trace h cancel out: the second, fourth, and fifth terms on the top line of the right-hand side sum to zero, as do the first, second, and fourth expressions on the bottom line of the right-hand side. What remains is

$$G_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha\partial^\mu\bar{h}_{\mu\beta} + \partial_\beta\partial^\mu\bar{h}_{\mu\alpha} - \eta_{\alpha\beta}\partial^\mu\partial^\nu\bar{h}_{\mu\nu} - \Box\bar{h}_{\alpha\beta}\right) \; . \tag{14.24}$$

It should be emphasized that this trick is really nothing more than algebraic sleight of hand; however, it is no less useful for being essentially a bit of clever rearranging.

Notice that in what remains all the terms that are left except the very last term inside the parentheses involve divergences of the metric perturbation: the first three terms are each of the form $\partial^\mu\bar{h}_{\mu\nu}$ (or something involving a different index than ν). Just as in electromagnetic theory, we can use our gauge freedom to clean this up. We would like to set

$$\partial^\mu\bar{h}_{\mu\nu} = 0 \; , \tag{14.25}$$

a total of 4 conditions. The quantities ξ_ν which generate the infinitesimal coordinate transformation provide us with 4 four free functions to do this. Writing

$$h^{\rm new}_{\mu\nu} = h^{\rm old}_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu \; , \tag{14.26}$$

we trace reverse and find that

$$\bar{h}^{\rm new}_{\mu\nu} = \bar{h}^{\rm old}_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu + \eta_{\mu\nu}\partial^\alpha\xi_\alpha \; . \tag{14.27}$$

Take the divergence of this:

$$\begin{aligned} \partial^\mu\bar{h}^{\rm new}_{\mu\nu} &= \partial^\mu\bar{h}^{\rm old}_{\mu\nu} - \Box\xi_\nu - \partial_\nu\partial^\mu\xi_\mu + \eta_{\mu\nu}\partial^\mu\partial^\alpha\xi_\alpha \\ &= \partial^\mu\bar{h}^{\rm old}_{\mu\nu} - \Box\xi_\nu \; . \end{aligned} \tag{14.28}$$

(To simplify, we used the metric to lower the index on the last term and recognized that μ and α are dummy indices on the final two terms on the top line.) This tells us that if we choose our gauge generators to satisfy

$$\Box\xi_\nu = \partial^\mu\bar{h}^{\rm old}_{\mu\nu} \; , \tag{14.29}$$

then the metric in our new gauge will be divergence free. Solutions of (14.29) always exist as long as $\bar{h}^{\rm old}_{\mu\nu}$ is well behaved, and this metric perturbation will always be well behaved in the weak gravity applications to which we apply this framework. We call this gauge "Lorenz gauge," following the essentially identical gauge choice used in electrodynamics to make the potential 4-vector divergence free. When we are in Lorenz gauge, the Einstein tensor is given by

$$G_{\alpha\beta} = -\frac{1}{2}\Box\bar{h}_{\alpha\beta} \; , \tag{14.30}$$

and we find the linearized Einstein field equation takes the form

$$\boxed{\boxed{\Box\bar{h}_{\alpha\beta} = -16\pi G T_{\alpha\beta}}} \tag{14.31}$$

This is starkly reminiscent of the sourced Maxwell's equation written in terms of the potential in Lorenz gauge, and very similar techniques can be used to construct its solution.

14.5 Static solutions

Let us begin by considering a perfect fluid source that is static ($u^t = 1 + \mathcal{O}(h)$, $u^i = 0$) and non-relativistic ($\rho \gg P$, corresponding to $c_{\rm sound} \ll c$). We can write the stress energy tensor

$$T_{\alpha\beta} \simeq \rho u_\alpha u_\beta \; . \tag{14.32}$$

Thanks to the static condition, the only non-negligible component is $T_{00} = \rho + \mathcal{O}(h)$. The only non-trivial field equation component is then

$$\Box\bar{h}_{00} = -16\pi G\rho \; . \tag{14.33}$$

Including the $\mathcal{O}(h)$ correction would lead to corrections to the solution $\bar{h}_{00}$ we find which are themselves second-order in h.

Since the source is static, the differential operator simplifies to

$$\nabla^2\bar{h}_{00} = -16\pi G\rho \; . \tag{14.34}$$

Solutions to this equation have a form reminiscent of a Coulomb potential. In the case of gravity, we recognize it as the equation for the Newtonian potential Φ (modulo a factor -4), and deduce that the solution is

$$\bar{h}_{00} = -4\Phi \; ; \qquad \text{all other } \bar{h}_{\mu\nu} = 0 \; . \tag{14.35}$$

To get the metric perturbation, we trace reverse again:

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h} \; , \tag{14.36}$$

where the trace $\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = \eta^{00}\bar{h}_{00} = (-1)(-4\Phi) = 4\Phi$. This tells us that

$$\begin{aligned} h_{00} &= -4\Phi - \frac{1}{2}(-1)(4\Phi) \\ &= -2\Phi \; , \end{aligned} \tag{14.37}$$

$$\begin{aligned} h_{11} = h_{22} = h_{33} &= 0 - \frac{1}{2}(1)(4\Phi) \\ &= -2\Phi \; ; \end{aligned} \tag{14.38}$$

all off-diagonal components of $h_{\mu\nu} = 0$ in this case. We see that

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2 + dy^2 + dz^2) \; . \tag{14.39}$$

This is the line element we used (without justification) to explore the Newtonian limit.

On problem set 7, you will examine what happens when the body spins rigidly about an axis. Just as the static problem leads to a potential that is a lot like the Coulomb potential governing the electric field arising from the density field ρ, the rotating problem looks a lot like what we get for the magnetic potential arising from a density flow $\mathbf{J} = \rho\mathbf{v}$. The solution should remind you quite a bit of problems you have already seen in an electromagnetic content, albeit with some signs flipped, and various prefactors not quite the same. It should also be noted that linearity of these field equations means that we can add any *homogeneous* solution of the field equation; i.e., any solution of (14.31) for which $T_{\alpha\beta} = 0$. Modulo some subtleties having to do with gauge, such solutions can represent radiation. Hold that thought! — we throw away those solutions right now because they are not static (our current focus), but will be a major focus of material we discuss very soon.