Last time, showed that the EFE for linearized gravity can be put into the form:

\[ \Box H_{\mu\nu} = -16\pi T_{\mu\nu} \]

If we put our coordinates into Lorentz gauge: \( \Box H_{\mu\nu} = 0 \).

For a static source, \( \Box \rightarrow \Box^2 = 8\pi \delta^{ij} \delta_{0j} \), simplifies considerably, not too hard to solve.

What about a non-static source? Use the fact that linear EFE is of a form commonly encountered in classical field theory:

\[ \Box f(t, x) = s(t, x) \]

Linearity means we can solve this using method of Green's functions [Unravel, 3rd ed., Sec. 16.5-16.6].

Replace \( s(t, x) \) with delta function, assert that the solution in this case is some function \( G \):

\[ \Box G(t, x; t', x') = \delta(t-t') \delta(x-x') \]

Replace \( s(t, x) \) with delta function:

\[ s(t, x) = \int dt' \int d^3 x' \ s(t', x') \delta(t-t') \delta(x-x') \]

\[ \rightarrow \ f(t, x) = \int dt' \int d^3 x' \ s(t', x') \ G(t, x; t', x') \]

gives \( \text{field permit} \) since.
The Green's function for the wave equation is well-known (and easily computed via Fourier analysis):

\[ G(t, \mathbf{x}; t', \mathbf{x}') = -\delta\left[t - (t - |\mathbf{x} - \mathbf{x}'|)\right] \frac{1}{4\pi |\mathbf{x} - \mathbf{x}'|} \]

(See, e.g., Jackson [2nd Ed], Sec. 6.6)

Notice the form \( t - |\mathbf{x} - \mathbf{x}'| \): This is "retarded time", the time at the field point minus the time it takes radiation to propagate from the source pt \( \mathbf{x}' \) to the field pt \( \mathbf{x} \).

![Diagram of Green's function concept]

Using this green's function, our Einstein solution is

\[ \bar{\Phi}(\mathbf{r}) = 4G \int d^3 x' \, \Theta\left(t - |\mathbf{x} - \mathbf{x}'|\right) G(t, \mathbf{x}; t', \mathbf{x}') \]

Mild problem: All components of the metric look indistinguishable!

This is because we choose a gauge which makes the field equation just a wave equation. Physical degrees of freedom will be masked by gauge effects!
Example: \[ A^0 = \frac{q}{r} - \frac{q \cos (kz - r - wt)}{r} \] \[ A^i = -\frac{q \cos (kz - r - wt)}{r} \frac{\partial \mathbf{E}}{\partial x} + \frac{q x \cos (kz - r - wt)}{r^3} - \frac{q x \sin (kz - r - wt)}{r} \frac{\partial \mathbf{E}}{\partial z} \]

Looks radiative. Construct field tensor:

\[ F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \]

\[ = \begin{bmatrix} 0 & -x & -y & -z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{bmatrix} \times \frac{q}{r^3} \]

\[ \rightarrow \mathbf{E}^i = \frac{q x^i}{r^3}, \quad \mathbf{B}^i = 0 \text{: Coulomb pt charge!} \]

What happened? Picked a stupid gauge:

\[ \lambda = \frac{q \cos (kz - r - wt)}{r} \]

\[ A^\mu \rightarrow A^\mu + \partial^\mu \lambda \]

Moral: Gauge can obscure physics if we're not careful.
Somewhat advanced topic: Recast metric & source in a form which allows us to categorize the radiative & non-radiative degrees of freedom of spacetime (in linearized theory, at least.)

End result: 4 physical degrees of freedom that are governed by \(\nabla^2\)-type equations

2 physical degrees of freedom governed by a wave equation: polarizations of GWs.

Extension by Flatman + Tye of idea developed by Ed Bertshinger

Consider \(h_{\mu\nu}\) as a tensor field on a flat background. Pick inertial coordinates; choose timelike & spacelike directions. The 10 components of \(h_{\mu\nu}\) then break into 3 subgroups when we consider how they behave under rotations:

\[h_{\mu\nu} \rightarrow h_{tt} = -2\Phi \quad \text{(scalar)}\]

\[h_{ti} \rightarrow \text{vector}\]

\[h_{ij} \rightarrow \text{tensor}\]

Break down further. Consider \(h_{ti}\): Any vector can be written as a divergence-free function plus the gradient of a scalar:

\[h_{ti} = \beta_i + \partial_i \chi, \quad \partial_i \beta_i = 0\]

Note: placement of spatial indices immaterial!
Next, consider $h_{ij}$. Extending our logic to a 2-index object, the most general form of this $3 \times 3$ symmetric tensor is

\[ h_{ij} = h_{ij}^{\text{TT}} + \frac{1}{3} H \delta_{ij} + \varepsilon \varepsilon_{ij} \]

\[ + (\varepsilon \varepsilon_{ij} - \frac{1}{3} \delta_{ij} \nabla^2) \lambda \]

(1) $H = \text{scalar}$. Notice: $\delta_{ij} h_{ij} = H$. This is the trace of $h_{ij}$.

(2) $\lambda = \text{scalar}$. This is the contribution to $h_{ij}$ that is the double gradient of a scalar. Notice that the divergence operator is traceless:

\[ \delta_{ij} \left( \varepsilon \varepsilon_{ij} - \frac{1}{3} \delta_{ij} \nabla^2 \right) = \nabla^2 - \frac{1}{3} \cdot 3 \nabla^2 = 0. \]

Smart: $H$ already has the trace.

(3-4) $\varepsilon_{ij} = \text{vector}$. Gives us contribution that is the gradient of a vector. In order that it have no trace, we require $\varepsilon \varepsilon_{ij} = 0$.

(6-3-4) $h_{ij}^{\text{TT}} = \text{tensor}$. Gives us the remaining divergence free, trace-free degrees of freedom:

\[ \delta_{ij} h_{ij}^{\text{TT}} = 0 \]

\[ \varepsilon \varepsilon_{ij} h_{ij}^{\text{TT}} = 0 \]

Total of 10 functions.
Goal: Develop EFE in terms of \((\phi, \beta^i, \gamma, H, \lambda, \varepsilon^i, h^{ij})\)

1st: examine gauge freedom. We can change the metric by

\[ h_{\mu\nu} \rightarrow h_{\mu\nu} - \partial_{[\mu} \xi_{\nu]} - \partial_{[\nu} \xi_{\mu]} \]

but leave all curvature unchanged.

Examine gauge generator: \(\xi^\mu = (\xi^t, \xi^i)\)

\[ \xi^\mu = (A, \beta^i + \partial^i \gamma) \text{ w/ } B^i = 0 \]

\(\nabla\) Grad. of scalar

\(\nabla\) Divergence-free vector

When we apply this to the metric, we find

\[ \phi \rightarrow \phi + \partial_t A \]
\[ \beta^i \rightarrow \beta^i - \partial_t B^i \]
\[ \gamma \rightarrow \gamma - A - \partial_t C \]
\[ H \rightarrow H - 2 \nabla^2 C \]
\[ \lambda \rightarrow \lambda - 2C \]
\[ \varepsilon^i \rightarrow \varepsilon^i - 2 \partial_t B^i \]

\(h^{ij} \rightarrow h^{ij} \leftarrow \text{gauge invariant D.O.F.!} \)
Stare at these; notice that the following combinations are gauge invariant:

$$\Phi = \phi + \partial_t \chi - \frac{1}{2} \partial^2 \lambda$$
$$\Theta = \frac{1}{3} (H - \nabla^2 \chi)$$
$$\Xi_i = \beta_i - \frac{1}{2} \epsilon_{ij} \partial^i \lambda$$

Only 6 functions! Of our original 10, only 6 are true physical degrees of freedom. Other 4 are pure gauge. Consistent with 10 Einstein eqs plus 4 constraints ($\nabla^2 \Phi + \rho = 0$).

Before apply this spacetime decomposition, need to decompose stress energy tensor:

$$T_{tt} = \delta$$
$$T_{ti} = S_i + \partial_i \delta$$
$$T_{ij} = \rho \delta_{ij} + \sigma_{ij} + \partial_i (\sigma_j) + \left( \partial_i \delta_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \sigma$$

$\delta$ = mass/energy density; $S_i$ = momentum density/energy flux.
$\rho$ = pressure; $\sigma_{ij}$ = anisotropic stress.

Constraints:
$$\partial_i S_i = 0 \quad \partial_i \sigma_i = 0$$
$$\partial_i \sigma_{ij} = 0 \quad \partial_j \sigma_{ij} = 0$$

No physics! Just rearranging terms.
Now, enforce $\partial^a T_{ap} = 0$; find:

\[ \nabla^2 S = \partial t S \]
\[ \nabla^2 \sigma = -\frac{3}{2} \mathcal{P} + \frac{3}{2} \partial t S \]
\[ \nabla^2 \sigma_i = 2 \partial t S_i \]

- Only $s, \mathcal{P}, S_i$, and $\sigma_{ij}$ are freely specifiable. These 6 functions determine the remaining 4 fields.

Now, grind:

\[ G_{tt} = -\nabla^2 \Theta \]
\[ G_{ti} = -\frac{1}{2} \nabla^2 \Phi_i - \partial_i \partial t \Theta \]
\[ \sigma_{ij} = -\frac{1}{3} \Box h_{ij} - \partial_i \partial_j (2 \Phi + \Theta) - \frac{1}{2} \partial_i \partial_j (2 \Phi + \Theta) - \partial t \Theta \]

Equate to source:

\[ \nabla^2 \Phi = 4\pi G \left( g + 3\mathcal{P} - 3\partial t S \right) \]
\[ \nabla^2 \Theta = -8\pi G g \]
\[ \nabla^2 \Phi_i = -16\pi G S_i \]
\[ \Box h_{ij} = -16\pi G \sigma_{ij} \]

**Remarkable cleanup!** 10 EFES reduce to 6 gauge invariant field equations. 4 are Poisson-type - yield solutions like Coulomb pt charge. Often called "longitudinal degrees of freedom". 2 are wave equations: "Radiative degrees of freedom".