MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF PHYSICS
8.962 SPRING 2024

LECTURE 15
SOLVING THE EINSTEIN FIELD EQUATIONS: DYNAMICAL WEAK FIELD SOLUTIONS

15.1 Solving the linearized Einstein equation for dynamical sources

When sources are not dynamical, then the linearized Einstein field equation

$$\Box\bar{h}_{\alpha\beta} = -16\pi G T_{\alpha\beta} \tag{15.1}$$

simplifies significantly: one can find a frame in which time derivatives on the left-hand side do not contribute (this is typically a frame in which a localized source is at rest in the coordinates), and so the wave operator $\Box$ simplifies to a Laplacian ∇^2. Such an equation can be solved using techniques imported directly from studies of Maxwell's equations. In particular, the leading order solution for non-relativistic sources ($\rho \gg P$) is just Newtonian gravity for a distribution of mass/energy density ρ, and can be solved as a multipolar expansion describing how ρ is distributed in space. Subleading solutions (such as you explore on problem set 7) correct the Newtonian description with gravitational contributions that describe how the flow of this mass/energy density contributes to spacetime. The entire problem bears a striking resemblance to how one solves for electric and magnetic fields from a stationary distribution of charges and currents.

How does one solve Eq. (15.1) when sources are dynamical? For example, what if one considers a source in which the mass/energy distribution is highly disturbed from equilibrium and undergoes oscillations? Or, a source consisting of two bodies in orbit about one another? There is no frame in which the time derivatives can be neglected in this case, and one must solve Eq. (15.1) in a manner which reflects the source's dynamics.

Our tool for beginning this investigation takes advantage of the fact that (15.1) is a linear equation — double the amount of source on the right-hand side, and we double[1] the amount of resulting field on the left-hand side. Linear differential equations can often be solved using the technique of Green's functions. If you have not seen this technique before, I recommend the discussion in the textbook by Arfken[2]. Briefly, imagine that you need to find the particular solution to a differential equation

$$\mathcal{D} f(t,\mathbf{x}) = s(t,\mathbf{x}) \;, \tag{15.2}$$

where $\mathcal{D}$ is a linear differential operator, $s(t,\mathbf{x})$ is a source that depends on space and time in the chosen coordinate system, and $f(t,\mathbf{x})$ is the resulting field arising from this source. The detailed form of the operator $\mathcal{D}$ is not important for this brief overview, other than that it must be linear in the field f.

The idea of the Green's function is to recognize that much of the challenge in this analysis comes from the fact that the source may be complicated. However, because the equation is linear, we can replace the source with something simpler and build a more complicated source using superposition. In particular, let us replace the source with delta functions:

$$s(t,\mathbf{x}) \to \delta(t-t')\delta^{(3)}(\mathbf{x}-\mathbf{x}') \;. \tag{15.3}$$

[1]This somewhat glib description neglects the role of homogeneous solutions to (15.1), i.e., solutions for no source at all, $\Box\bar{h}_{\alpha\beta} = 0$. The amount of homogeneous solution in the problem is determined by boundary conditions; though very important to get this right, we can set this aside for the moment.

[2]G. B. Arfken, *Mathematical Methods for Physicists*, Sec. 16.5–16.6 in the 3rd edition. Apparently this book is now up to edition 7; the location of the discussion of Green's functions may be different in newer versions.

The primed coordinates $(t', \mathbf{x}')$ label the "source point," the location of the delta source. Our goal is to compute how much field arises at field point $(t, \mathbf{x})$ arisesdue to this delta spike of source at the source point. We call this solution the Green's function $G(t, \mathbf{x}; t', \mathbf{x}')$, and assert that our equation can now be written

$$\mathcal{D}G(t, \mathbf{x}; t', \mathbf{x}') = \delta(t - t')\delta^{(3)}(\mathbf{x} - \mathbf{x}') \ . \tag{15.4}$$

Suppose we have constructed the Green's function corresponding to our operator $\mathcal{D}$. It should be emphasized that this construction may take some effort; but imagine we have put in this effort, and we have $G(t, \mathbf{x}; t', \mathbf{x}')$ in hand for our problem. The solution for our field is then given by

$$f(t, \mathbf{x}) = \int dt' \int d^3x' G(t, \mathbf{x}; t', \mathbf{x}') s(t', \mathbf{x}') \ . \tag{15.5}$$

We can think of the Green's function $G(t, \mathbf{x}; t', \mathbf{x}')$ as providing the amount of field at $(t, \mathbf{x})$ that arises per unit of source at $(t', \mathbf{x}')$. This solution is essentially a convolution: we treat the source as a superposition of many delta functions, so by linearity the resulting field is a superposition of many Green's functions.

To prove that Eq. (15.5) gives the correct solution, simply act on it using the operator $\mathcal{D}$:

$$\begin{aligned}
\mathcal{D}f(t, \mathbf{x}) &= \int dt' \int d^3x' \mathcal{D}G(t, \mathbf{x}; t', \mathbf{x}') s(t', \mathbf{x}') \\
&= \int dt' \int d^3x' \delta(t - t')\delta^{(3)}(\mathbf{x} - \mathbf{x}') s(t', \mathbf{x}') \\
&= s(t, \mathbf{x}) \ .
\end{aligned} \tag{15.6}$$

On the first line, we use the fact that the operator $\mathcal{D}$ only acts on functions of the unprimed coordinates, so it commutes with integration over the primed coordinates, and thus hits the Green's function but not the source. We then use the definition of the Green's function, and are left with an integral that is trivial to perform. Thus, if we can develop the Green's function for our differential operator, then we can easily construct the particular solution that solves our equation.

Fortunately, the Green's function for the wave equation is very well known: the solution to

$$\Box G(t, \mathbf{x}; t', \mathbf{x}') = \delta(t - t')\delta^{(3)}(\mathbf{x} - \mathbf{x}') \tag{15.7}$$

is

$$G(t, \mathbf{x}; t', \mathbf{x}') = -\frac{\delta\left[t' - (t - |\mathbf{x} - \mathbf{x}'|)\right]}{4\pi|\mathbf{x} - \mathbf{x}'|} \ . \tag{15.8}$$

This *radiative Green's function* is derived in detail in, for example, J. D. Jackson, *Classical Electrodynamics*, Sec. 6.6 (2nd edition); we provide a sketch of this derivation in the next section of these notes. Notice the form

$$t - |\mathbf{x} - \mathbf{x}'| \equiv t_{\rm ret} \tag{15.9}$$

that enters the time dependence. The name $t_{\rm ret}$ we assign means that this is *retarded time*: it reflects the fact that information about the source at time t' and location $\mathbf{x}'$ only reaches an observer at $\mathbf{x}$ following an interval $|\mathbf{x} - \mathbf{x}'|$ (remember our $c = 1$ units). Our wave operator[3] describes functions which propagate information about sources across spacetime at the speed of light.

The general solution to Eq. (15.1) is thus given by

$$\begin{aligned}
\bar{h}_{\alpha\beta}(t, \mathbf{x}) &= \int dt' \int d^3x' \left(-\frac{\delta\left[t' - (t - |\mathbf{x} - \mathbf{x}'|)\right]}{4\pi|\mathbf{x} - \mathbf{x}'|}\right)\left(-16\pi G T_{\alpha\beta}(t', \mathbf{x}')\right) \\
&= 4G \int d^3x' \frac{T_{\alpha\beta}(t_{\rm ret}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \ .
\end{aligned} \tag{15.10}$$

[3]Because the wave operator is second order in time derivatives, it is agnostic about whether this information propagates forward or backwards in time; both solutions work with the ∂_t^2 part of the $\Box$ operator. We discard an *advanced* solution that depends on $t_{\rm adv} = t + |\mathbf{x} - \mathbf{x}'|$ on the grounds that information propagating from the future to the past is inconsistent with the ordering of events. In essence, we enforce a boundary condition that information can only propagate from the past to the future.

15.2 Aside: The radiative Green's function

Our goal will be to use the solution (15.10) to characterize and understand solutions to the linearized Einstein equation. Before doing this, we briefly discuss where the radiative Green's function comes from, for the benefit of students who have not yet encountered this in their coursework.

We need two ingredients for this analysis:

- The Laplace operator acting on $1/|\mathbf{x}-\mathbf{x}'|$ generates the Dirac delta function:

$$\nabla^2\left(\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right) = -4\pi\,\delta^{(3)}(\mathbf{x}-\mathbf{x}')\ . \tag{15.11}$$

  This relationship is discussed in many electrodynamics textbooks; the discussion in Griffiths[4] is typically clear and readable if you have not seen this result previously.

- The Fourier transform pair: if $f(t)$ is some function of time, then we define its Fourier transform

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt\ . \tag{15.12}$$

  Note that this implies that the Fourier transform of $df/dt = i\omega\tilde{f}(\omega)$. The Fourier transform operation is inverted by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\omega)e^{-i\omega t}d\omega\ . \tag{15.13}$$

  By combining the forward and reverse Fourier transforms, one can show that

$$\delta(t'-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t'-t)}d\omega\ . \tag{15.14}$$

We begin by computing the Green's function for the *inhomogeneous Helmholtz equation*:

$$\left(\nabla^2 + k^2\right)G(\mathbf{x};\mathbf{x}') = \delta^{(3)}(\mathbf{x}-\mathbf{x}')\ . \tag{15.15}$$

For notational simplicity, let us define $\boldsymbol{\zeta} \equiv \mathbf{x}-\mathbf{x}'$, $\zeta = |\mathbf{x}-\mathbf{x}'|$. The Green's function which solves this can only depend on ζ, and must be spherically symmetric about $\boldsymbol{\zeta} = 0$. This allows us to rewrite this equation as

$$\frac{1}{\zeta}\frac{d^2}{d\zeta^2}(\zeta G) + k^2 G = \delta^{(3)}(\boldsymbol{\zeta})\ . \tag{15.16}$$

This form of the equation makes it clear that

$$G = -\frac{e^{\pm ik\zeta}}{4\pi\zeta}\ . \tag{15.17}$$

That this solution holds is clear for $\zeta > 0$. Exactly at $\zeta = 0$, we invoke the first Dirac delta identity and see that the solutions are equivalent at this point[5]. Returning to the original parameterization, we see that

$$\left(\nabla^2 + k^2\right)G(\mathbf{x};\mathbf{x}') = \delta^{(3)}(\mathbf{x}-\mathbf{x}') \tag{15.18}$$

is solved by

$$G(\mathbf{x};\mathbf{x}') = -\frac{e^{\pm ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|}\ . \tag{15.19}$$

[4] D. J. Griffiths, *Introduction to Electrodynamics*, Sec. 1.5.3, 4th edition.

[5] A more careful analysis would examine a very small region surrounding $\boldsymbol{\zeta} = 0$ and show that integrating the left-hand side of (15.16) over such a region returns a non-zero value consistent with with integrating the right-hand side of this equation over the same region. Griffith's discussion of Eq. (15.11) is an excellent prototype of such a calculation.

Note the ambiguous sign in the argument of the exponential.

Turn now to the wave equation:

$$\Box G(t, \mathbf{x}; t', \mathbf{x}') = (-\partial_t^2 + \nabla^2) G = \delta(t - t')\delta^{(3)}(\mathbf{x} - \mathbf{x}') . \tag{15.20}$$

Take the Fourier transform of both sides. On the left-hand side, $G(t, \mathbf{x}; t', \mathbf{x}') \to \tilde{G}(\omega, \mathbf{x}; t', \mathbf{x}')$, $\partial_t^2 G \to -\omega^2 \tilde{G}$; on the right-hand side, $\delta(t - t') \to e^{i\omega t'}$. The equation becomes

$$\left(\nabla^2 + \omega^2\right) \tilde{G} = e^{i\omega t'} \delta^{(3)}(\mathbf{x} - \mathbf{x}') . \tag{15.21}$$

This is the inhomogeneous Helmholtz equation with $k = \omega$, and with an extra factor of $e^{i\omega t'}$ attached to the right-hand side. We can simply lift the solution from our earlier discussion, finding

$$\tilde{G}(\omega, \mathbf{x}; t', \mathbf{x}') = -\frac{e^{i\omega(t' \pm |\mathbf{x} - \mathbf{x}'|)}}{4\pi|\mathbf{x} - \mathbf{x}'|} . \tag{15.22}$$

We conclude by computing the inverse Fourier transform:

$$\begin{aligned}
G(t, \mathbf{x}; t', \mathbf{x}') &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \tilde{G}(\omega, \mathbf{x}; t', \mathbf{x}') \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, e^{-i\omega t} \left[-\frac{e^{i\omega(t' \pm |\mathbf{x} - \mathbf{x}'|)}}{4\pi|\mathbf{x} - \mathbf{x}'|} \right] \\
&= -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}'|} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left[i\omega \left(t' - \left(t \mp |\mathbf{x} - \mathbf{x}'| \right) \right) \right] d\omega \right\} \\
&= -\frac{\delta\left[t' - (t \mp |\mathbf{x} - \mathbf{x}'|)\right]}{4\pi|\mathbf{x} - \mathbf{x}'|} \\
&= -\frac{\delta\left(t' - t_{\mathrm{ret/adv}}\right)}{4\pi|\mathbf{x} - \mathbf{x}'|} .
\end{aligned} \tag{15.23}$$

The advanced solution, $t' = t_{\mathrm{adv}} = t + |\mathbf{x} - \mathbf{x}'|$, describes information propagating backwards in time. It arises, as mentioned in a footnote on a previous page, because the wave equation, being second order in time, is agnostic about whether information propagates from the past to the future or vice versa. We are not agnostic about time's direction, and enforce the causal boundary condition by rejecting the advanced solution. The radiative Green's function we use is thus given by

$$\boxed{\boxed{G(t, \mathbf{x}; t', \mathbf{x}') = -\frac{\delta\left(t' - t_{\mathrm{ret}}\right)}{4\pi|\mathbf{x} - \mathbf{x}'|}}} \tag{15.24}$$

where $t_{\mathrm{ret}} = t - |\mathbf{x} - \mathbf{x}'|$.

15.3 The general solution of the linearized Einstein field equation

It should be emphasized that the solution we have derived to the linearized Einstein field equation,

$$\bar{h}_{\alpha\beta}(t, \mathbf{x}) = 4G \int d^3x' \frac{T_{\alpha\beta}(t_{\mathrm{ret}}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} , \tag{15.25}$$

is *exact*, at least within the confines of the linearized approximation to general relativity. We made a specific gauge choice in order to write the field equation in a form which is amenable to our use of the radiative Green's function. This does not limit our solution in the slightest. Equation (15.25) can be used to find the spacetime for any stress-energy tensor, provided the resulting deviation from flat spacetime is weak enough that h^2 terms can be neglected.

This solution has an interesting character: the field at event $(t, \mathbf{x})$ depends upon the source at $(t_{\rm ret}, \mathbf{x}')$, meaning that the dynamics enters our solution with a retardation factor that reflects the finite speed with which information propagates. And, the solution has a factor that reflects a "one over distance" kind of falloff. These are exactly the behaviors that describe radiative potentials in electrodynamics. It looks like all ten components of $\bar{h}_{\alpha\beta}$ describe a radiative solution that propagates out from a source!

This might initially seem like a very interesting and physically reasonable outcome, but with a little thought you should be concerned. Suppose some aspect of the source reflects a static mass M centered at $\mathbf{x}'$. We expect that to generate metric components with contributions proportional to $GM/|\mathbf{x} - \mathbf{x}'|$, which are not radiative.

For some intuition about what might be happening here, consider an electromagnetic potential whose components in some frame are given by

$$\begin{aligned} A^0 &= \frac{q}{r} + \frac{q\omega R \sin(\mathbf{k}\cdot\mathbf{x} - \omega t)}{r} , \\ A^i &= \frac{qk^i R \sin(\mathbf{k}\cdot\mathbf{x} - \omega t)}{r} + \frac{qx^i R \cos(\mathbf{k}\cdot\mathbf{x} - \omega t)}{r^3} . \end{aligned} \tag{15.26}$$

If, while walking down a dark alley, someone shoved this potential in your face and demanded that you describe what physical situation this corresponded to, you would probably guess that it is a Coulomb point charge plus a plane electromagnetic wave. Once you are safe from the physicist mugger, you would want to compute the components of the field tensor. Doing so, you discover[6] the components have the following values:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -x & -y & -z \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ z & 0 & 0 & 0 \end{pmatrix} \times \frac{q}{r^3} . \tag{15.27}$$

This potential describes an electric field $\mathbf{E} = q\mathbf{x}/r^3$ and a magnetic field $\mathbf{B} = 0$: it is simply a Coulomb point charge at the origin. There is *no* radiation in this problem at all.

The potential that the mugger shoved in your face was nothing but the usual Coulomb potential, written in a gauge that made it seem radiative. Indeed, to generate this, I started with $A^0 = q/r$, $A^i = 0$ and applied a gauge change $A^\mu \to A^\mu - \partial^\mu \Lambda$ with $\Lambda = qR\cos(\mathbf{k}\cdot\mathbf{x} - \omega t)/r$.

15.4 Gauge invariant characterization of linearized theory

The moral of this story is that when we work with quantities like the metric, the gauge we use can obscure the physics if we are not careful. This motivates a way of understanding which aspects of a spacetime are truly radiative in all gauges, and which are not (at least when we linearize around a flat background spacetime). The following discussion is a brief synopsis of a framework that was presented in E. E. Flanagan and S. A. Hughes, gr-qc/0501041 (available at `arXiv.org`, and linked to the 8.962 website); this presentation in turn was based on ideas we learned from E. Bertschinger, which performed a similar analysis to perturbations of cosmological spacetimes. The end result is that we will find that the degrees of freedom represented by the 10 independent components of the spacetime perturbation $h_{\alpha\beta}$ describe 4 degrees of freedom governed by Poisson-like equations (and thus are not radiative) and 2 degrees of freedom governed by a wave equation. These represent 2 polarizations of gravitational radiation, and will be the main topic of our next lecture. (The remaining 4 degrees of freedom are purely gauge in nature.)

Begin by considering $h_{\mu\nu}$ as a tensor field in a flat background spacetime. We pick inertial coordinates, meaning that we select a time coordinate that we will focus on for this analysis, as

[6] If you check this, don't forget that $\partial^t = \eta^{00}\partial_t = -\partial/\partial t$.

well as 3 space directions. The 10 components of $h_{\mu\nu}$ then break up into 3 subgroups when we consider how they behave under rotations:

$$\begin{aligned} h_{\mu\nu} \quad \mapsto \quad & h_{tt} \equiv -2\phi \qquad \text{(a scalar)} \\ & h_{ti} \qquad \text{(a 3-vector)} \\ & h_{ij} \qquad \text{(a } 3\times 3 \text{ symmetric tensor)}\ . \end{aligned} \tag{15.28}$$

We can break these down further. Consider h_{ti} first. Any 3-vector can be written as a divergence-free function plus the gradient of a scalar, so we write

$$h_{ti} = \beta_i + \partial_i\gamma\ , \qquad \partial_i\beta_i = 0\ . \tag{15.29}$$

This represents a total 3 degrees of freedom (1 for the function γ, 3 for the functions β_i, plus one constraint imposed by $\partial_i\beta_i = 0$). Note that in this analysis, the placement of spatial indices is immaterial — our time coordinate is fixed, so the spatial metric is just the Kronecker delta. (We modify the summation convention in this situation to sum over repeated indices no matter how they are positioned.) Including the free function ϕ, we have accounted for 4 degrees of freedom in the spacetime metric so far.

Extending this logic to h_{ij}, we find that the most general form of this 3×3 symmetric tensor is

$$h_{ij} = h_{ij}^{\rm TT} + \frac{1}{3}H\delta_{ij} + \partial_{(i}\epsilon_{j)} + (\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2)\lambda\ . \tag{15.30}$$

Let's go through the quantities we have introduced here and examine their properties:

- The function H is a scalar under rotations, and represents 1 degree of freedom. We put $H = \delta^{ij}h_{ij}$, so that it is the trace of this 3×3 tensor. This means there is no contribution to the trace from any other terms in Eq. (15.30).

- The function λ is also a scalar under rotations, and represents 1 degree of freedom. It represents a trace-free double gradient of a scalar. The operator which acts upon it is defined to have zero trace since the trace is already bound up in the function H.

- The functions ϵ_j behave as a 3-vector under rotations. In order that it have no trace, we require that $\partial_i\epsilon_i = 0$. This means that this quantity represents 3 free functions plus 1 constraint, for a total of 2 degrees of freedom.

- The functions $h_{ij}^{\rm TT}$ describe the remaining divergence-free, trace-free degrees of freedom in h_{ij}. The 6 functions of $h_{ij}^{\rm TT}$ are subject to 3 constraints to make this quantity divergence free, $\partial_i h_{ij}^{\rm TT} = 0$, plus 1 constraint to make it trace free, $\delta^{ij}h_{ij}^{\rm TT} = 0$. These functions thus represent 2 degrees of freedom. The label "TT" indicates that these represent the *transverse* and *trace-free* parts of h_{ij}. "Trace-free" is hopefully clear; the reason why divergenceless corresponds to transverse will be made clear in the next lecture.

These choices sum to a total of 6 degrees of freedom. Including the 3 degrees of freedom associated with γ and β_i and the 1 associated with ϕ, we see that all 10 components of $h_{\mu\nu}$ are accounted for in the degrees of freedom provided by this decomposition.

Before we start working with this metric, we should account for our gauge freedom: we can change the metric by $h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu$. Doing this has no effect on the curvature tensors, and thus no effect on the Einstein field equations. We write our gauge generator

$$\begin{aligned} \xi_\alpha &\doteq (\xi_t, \xi_i) \\ &\doteq (A, B_i + \partial_i C) \qquad \text{with} \quad \partial_i B_i = 0\ . \end{aligned} \tag{15.31}$$

The functions A and C are both scalars; B_i is a divergence-free vector. When we use this form of the generator to change gauge, we find that the functions we introduced to decompose the metric change as follows:

$$\begin{aligned}
\phi &\to \phi + \partial_t A \;, \\
\beta_i &\to \beta_i - \partial_t B_i \;, \\
\gamma &\to \gamma - A - \partial_t C \;, \\
H &\to H - 2\nabla^2 C \;, \\
\lambda &\to \lambda - 2C \;, \\
\epsilon_i &\to \epsilon_i - 2B_i \;, \\
h_{ij}^{\rm TT} &\to h_{ij}^{\rm TT} \;.
\end{aligned} \tag{15.32}$$

Notice that the TT contribution to h_{ij} is gauge invariant; these two degrees of freedom have the same form in *all* gauges. If we "stare" at these functions for a little while, we "notice" that the following combinations of metric functions are totally gauge invariant[7]:

$$\begin{aligned}
\Phi &= \phi + \partial_t\gamma - \frac{1}{2}\partial_t^2\lambda \;, \\
\Theta &= \frac{1}{3}\left(H - \nabla^2\lambda\right) \;, \\
\Psi_i &= \beta_i - \frac{1}{2}\partial_t\epsilon_i \;, \quad \partial_i\Psi_i = 0 \;.
\end{aligned} \tag{15.33}$$

These quantities, plus $h_{ij}^{\rm TT}$, represent the gauge-invariant, physical degrees of freedom in the metric $h_{\mu\nu}$. Notice that there are only 6 such degrees of freedom; in the 10 independent components of $h_{\mu\nu}$, 4 are purely gauge degrees of freedom. As we'll see when we talk in more depth about the dynamics of the Einstein field equations in general, this is consistent with the fact that the 10 Einstein field equations ($G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$) are subject to 4 constraints due to the contracted Bianchi identity ($\nabla^\alpha G_{\alpha\beta} = 0$).

We would now like to see what this decomposition of spacetime implies for the Einstein field equations. Because, as shown in Lecture 14, curvature tensor components do not depend on the gauge choice in linearized theory, we should be able to write any curvature tensor using only the gauge-invariant functions Φ, Θ, Ψ_i, and $h_{ij}^{\rm TT}$. We will take advantage of this fact to write out the components of the Einstein tensor in this form. Before doing so, we first introduce a decomposition of the stress-energy tensor similar to what we used in the metric:

$$\begin{aligned}
T_{tt} &= \rho \;, \\
T_{ti} &= S_i + \partial_i S \;, \\
T_{ij} &= P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\nabla^2\right)\sigma \;.
\end{aligned} \tag{15.34}$$

These quantities are subject to the constraints

$$\begin{aligned}
\partial_i S_i = 0 \;, \qquad & \partial_i\sigma_i = 0 \;, \\
\partial_i\sigma_{ij} = 0 \;, \qquad & \delta^{ij}\sigma_{ij} = 0 \;.
\end{aligned} \tag{15.35}$$

Several of these terms we have introduced have clear physical meaning: ρ is an energy density, P is a pressure, σ_{ij} is an anisotropic stress. However, some of these are merely rearranging terms in

[7]Scare quotes acknowledging that "stare" and "notice" somewhat understate the challenge of this process. However, it really is just a matter of gathering together functions in such a way that the pieces A, B_i, C which go into the shifted metric functions do not appear in the combinations that we assemble.

a way that is convenient. For instance, T_{ti} is a the flow of energy; $S_i + \partial_i S$ just rewrites this flow using functions with mathematically convenient properties. When we enforce $\nabla^\alpha T_{\alpha\beta} = 0$, we find that

$$\begin{aligned}
\nabla^2 S &= \partial_t \rho \;, \\
\nabla^2 \sigma &= -\frac{3}{2} P + \frac{3}{2}\partial_t S \;, \\
\nabla^2 \sigma_i &= 2\partial_t S_i \;.
\end{aligned} \tag{15.36}$$

This amplifies the message that some of the quantities we introduced in the decomposition of the stress-energy tensor are not independent. We can freely specify ρ, P, S_i, and σ_{ij}, a total of 6 functions describing the source; the remaining 4 functions S, σ, and σ_i which complete the source are determined by the behavior of the 6 we can freely specify.

With some labor, we develop the ten components of the Einstein tensor and write the results using the gauge-invariant metric functions we worked out. The result of this exercise is

$$\begin{aligned}
G_{tt} &= -\nabla^2 \Theta \;, \\
G_{ti} &= -\frac{1}{2}\nabla^2 \Psi_i - \partial_i \partial_t \Theta \;, \\
G_{ij} &= -\frac{1}{2}\Box h^{\rm TT}_{ij} - \partial_{(i}\Psi_{j)} - \frac{1}{2}\partial_i\partial_j \left(2\Phi + \Theta\right) + \delta_{ij}\left(\frac{1}{2}\nabla^2(2\Phi + \Theta) - \partial_t^2 \Theta\right) \;.
\end{aligned} \tag{15.37}$$

Enforcing $G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$ yields the following 6 differential equations governing the gauge-invariant degrees of freedom in spacetime:

$$\nabla^2 \Phi = 4\pi G \left(\rho + 3P - \partial_t S\right) \;, \tag{15.38}$$

$$\nabla^2 \Theta = -8\pi G \rho \;, \tag{15.39}$$

$$\nabla^2 \Psi_i = -16\pi G S_i \;, \tag{15.40}$$

$$\Box h^{\rm TT}_{ij} = -16\pi G \sigma_{ij} \;. \tag{15.41}$$

This exercise confirms that, although the solution we derived earlier appears to be one in which the metric perturbation is entirely radiative, this is quite misleading. No more than *two* components of the spacetime — those described by the transverse, traceless piece of the metric, $h^{\rm TT}_{ij}$ — satisfy a wave equation in all gauges. Therefore, only those two components truly characterize radiative components of the spacetime. Four of the remaining components of the spacetime are governed by Poisson-type equations, and are often called the "longitudinal degrees of freedom" associated with spacetime. The last four components represent gauge degrees of freedom.