Massachusetts Institute of Technology
Department of Physics
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Lecture 16
Gravitational radiation

16.1 Lesson of the gauge-invariant characterization of metric components

In the previous lecture, we showed that at linear order in deviations from flat spacetime, the Einstein field equations can be reduced to a single wave equation plus several Poisson-type equations:

$$\nabla^2\Phi = 4\pi G\left(\rho + 3P - \partial_t S\right) , \tag{16.1}$$
$$\nabla^2\Theta = -8\pi G\rho , \tag{16.2}$$
$$\nabla^2\Psi_i = -16\pi G S_i , \tag{16.3}$$
$$\Box h^{\rm TT}_{ij} = -16\pi G\sigma_{ij} . \tag{16.4}$$

See the previous lecture for precise definitions connecting the fields Φ, Θ, and Ψ_i to the metric $h_{\alpha\beta}$, and to see how the source terms that appear on the right-hand side of these equations connect to the stress-energy tensor components $T_{\alpha\beta}$.

Our focus in this lecture is to understand the nature of the radiative part of this solution. The transverse, traceless spatial metric components $h^{\rm TT}_{ij}$ describe a propagating, radiative spacetime contribution in all gauges, and thus truly describe a radiative component to the gravitational interaction. Radiation is a starkly non-Newtonian aspect of relativistic gravity, but an absolutely necessary one: *any* relativistic field theory must include radiation in order to connect distant fields to the behavior of a dynamical source, and this radiation must propagate with finite speed. Our particular goals are to understand the nature of $h^{\rm TT}_{ij}$ in terms of quantities that we can observe, and to see how one computes $h^{\rm TT}_{ij}$ given a source.

16.2 A purely radiative solution

To begin, we simply write down a metric that has $\Phi = \Theta = \Psi_i = 0$, and choose $h^{\rm TT}_{ij} = h^{\rm TT}_{ij}(t-z)$ — radiation propagating in the z direction. Our solution takes the form

$$h_{\alpha\beta} \doteq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{\rm TT}_{xx} & h^{\rm TT}_{xy} & 0 \\ 0 & h^{\rm TT}_{yx} & h^{\rm TT}_{yy} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{16.5}$$

Because this must be symmetric, $h^{\rm TT}_{xy} = h^{\rm TT}_{yx}$; because we demand that it be trace-free, $\delta^{ij}h^{\rm TT}_{ij} = 0$, we have $h^{\rm TT}_{xx} = -h^{\rm TT}_{yy}$. We will return to other aspects of this solution (e.g., why all components of the form $h_{\alpha z}$ are zero) later in this lecture.

What do freely falling bodies experience in this spacetime? Freely falling bodies follow geodesics, so we naturally start by looking at geodesics in the spacetime (16.5): we examine

$$\frac{du^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu}u^\mu u^\nu = 0 , \tag{16.6}$$

taking the bodies to be initially at rest in these coordinates, $u^\alpha(\tau = 0) \doteq (1, 0, 0, 0)$, and computing the connection coefficient from (16.5):

$$\begin{aligned}
\Gamma^\alpha{}_{00} &= \eta^{\alpha\beta}\Gamma_{\beta 00} \\
&= \frac{1}{2}\eta^{\alpha\beta}\left(\partial_t h^{\rm TT}_{\beta 0} + \partial_t h^{\rm TT}_{0\beta} - \partial_\beta h^{\rm TT}_{00}\right) \\
&= 0\,. \qquad (16.7)
\end{aligned}$$

This means that $du^\alpha/d\tau = 0$ at $\tau = 0$; integrating this up, we find that the body has $u^\alpha \dot{(}1, 0, 0, 0)$ for all time.

Does this mean that this radiative solution has no effect? No! — what it means is that we've calculated the wrong thing. Geodesics tell us about the trajectory of a body through a specified coordinate system. We see that a body remains at rest *with respect to these coordinates* as the radiation passes by. But we have tremendous freedom in general relativity to adjust our coordinates, and it could very well be that the coordinates are themselves moving in some way.

Indeed, we know that we can make the metric have a representation that looks just like special relativity in the vicinity of any event. To detect an effect involving the radiation encoded in Eq. (16.5), we should examine some kind of *proper* quantity — i.e., something that we can measure, with value that is independent of the representation — and it should be something that involves enough of a spatial extent that it can "taste" the curvature corrections which provide the leading correction to the $\eta_{\alpha\beta}$ representation of the metric in a local Lorentz frame. Let us examine two bodies that are separated by a displacement L in the x direction:

Let's compute the time takes for a light pulse to travel from A to B. The light pulse follows a null geodesic, so

$$0 = -\left(\frac{dt}{d\lambda}\right)^2 + (1 + h^{\rm TT}_{xx})\left(\frac{dx}{d\lambda}\right)^2 . \qquad (16.8)$$

Rearranging, this gives us an equation for how much time dt elapses as the light travels through a spatial interval dx:

$$\begin{aligned}
dt &= \sqrt{1 + h^{\rm TT}_{xx}}\,dx \\
&\simeq \left(1 + \frac{h^{\rm TT}_{xx}}{2}\right) dx\,. \qquad (16.9)
\end{aligned}$$

We've taken the positive root because light pulse travels in the direction of increasing x. If we examined light traveling in the opposite direction, or if the light reflected at $x = L$ and returning to $x = 0$, we would take the opposite sign. On the second line, we've used the fact that all the elements of $h_{\alpha\beta}$ are small (we are using linearized theory, after all) and so we use the binomial expansion. Integrating this up using the fact that A and B are located at fixed coordinate values $x = 0$ and $x = L$ respectively, the time it takes for light to travel from A to B is

$$\begin{aligned}
T_{A\to B} &= \int_0^L \left(1 + \frac{h^{\rm TT}_{xx}}{2}\right) dx \\
&= L + \frac{1}{2}\int_0^L h^{\rm TT}_{xx}\,dx\,. \qquad (16.10)
\end{aligned}$$

This is enough to show us that the time of arrival of light pulses varies with the amplitude of this component of the radiative solution $h^{\rm TT}_{xx}$. By making use of a very well-defined time standard (and monochromatic light itself defines such a time standard), Eq. (16.10) defines a way in which the imprint of such *gravitational radiation* or *gravitational waves* can be measured.

Note that $h^{\rm TT}_{xx}$ is a function of t and z. As such, one might imagine that it pops out of the integral. However, an integral with respect to x is in a sense also an integral with respect to t, since the light is traveling along a trajectory with $dx/dt = 1 + \mathcal{O}(h)$. Because of this, the function $h^{\rm TT}_{xx}$ in general is not constant inside the integral. This means that without more information about the nature of the wave amplitude $h^{\rm TT}_{xx}$, we cannot simplify (16.10) further. However, there are some important circumstances in which further manipulation can be done; you will explore such a circumstance on problem set 8.

16.3 Geodesic deviation and gravitational radiation

Both body A and body B are freely falling bodies that follow geodesics. Because they are spatially separated from one another in the radiative spacetime (16.5), free fall for body A is a little different from free fall for body B. This shows up in the fact that the proper separation, as measured by light travel time, changes proportional to the amplitude of the radiation. A nice way to describe two nearby geodesics is to examine their geodesic separation. Let us take the two bodies to be close enough that, at least initially, they have the same 4-velocity:

$$u^\alpha_A \simeq u^\alpha_B \doteq (1, 0, 0, 0) + \mathcal{O}(h) \; . \tag{16.11}$$

Slight differences at the level of h in their 4-velocities will couple to produce $\mathcal{O}(h^2)$ differences in the analysis that follows, which we can neglect. Taking the separation vector of these two bodies to be given by Y^α, the equation of geodesic deviation is given by

$$\frac{D^2 Y^\alpha}{d\tau^2} = R^\alpha{}_{\mu\nu\beta} u^\mu u^\nu Y^\beta \; . \tag{16.12}$$

Inserting our 4-velocities, expanding the various derivatives, and using one of the Riemann symmetries, this becomes

$$\frac{d^2 Y^i}{dt^2} = -R^i{}_{0j0} Y^j + \mathcal{O}(h^2) \; . \tag{16.13}$$

For our radiative perturbation, the only unique non-zero Riemann components are

$$R^x{}_{0x0} = -\frac{1}{2}\partial_t^2 h^{\rm TT}_{xx} \; , \tag{16.14}$$

$$\begin{aligned} R^y{}_{0y0} &= -\frac{1}{2}\partial_t^2 h^{\rm TT}_{yy} \\ &= \frac{1}{2}\partial_t^2 h^{\rm TT}_{xx} \; , \end{aligned} \tag{16.15}$$

$$\begin{aligned} R^y{}_{0x0} &= -\frac{1}{2}\partial_t^2 h^{\rm TT}_{xy} \\ &= R^x{}_{0y0} \end{aligned} \tag{16.16}$$

All other components are either zero, or are related by either a Riemann symmetry or the fact that $\partial_z = -\partial_z$ when it acts on the metric (thanks to the fact that the metric depends on $t - z$). Using these forms, the equation of geodesic deviation becomes

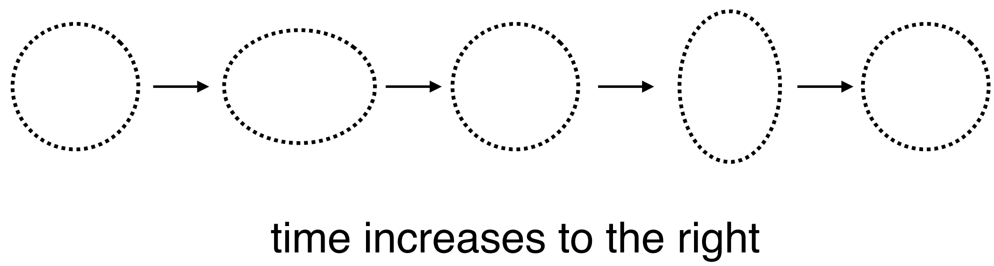
$$\frac{d^2 Y^x}{dt^2} = \frac{1}{2}\partial_t^2 h^{\rm TT}_{xx} Y^x + \frac{1}{2}\partial_t^2 h^{\rm TT}_{xy} Y^y \tag{16.17}$$

$$\frac{d^2 Y^y}{dt^2} = \frac{1}{2}\partial_t^2 h^{\rm TT}_{xy} Y^x - \frac{1}{2}\partial_t^2 h^{\rm TT}_{xx} Y^y \tag{16.18}$$

$$\frac{d^2 Y^z}{dt^2} = 0 \; . \tag{16.19}$$

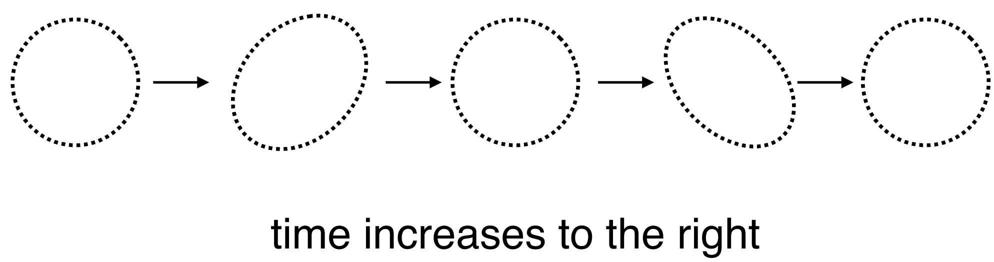
Let's assume that $Y^i = Y^i_0 + \delta Y^i$, with Y^i_0 a constant and δY^i having a magnitude $\mathcal{O}(h)$. Consider 2 limits:

- First, imagine that $h^{\rm TT}_{xx}$ is a sinusoid, and $h^{\rm TT}_{xy} = 0$. Imagine a ring of test masses in the (x, y) plane. As this gravitational wave passes by, the separation vector components $(\delta Y^x, \delta Y^y)$ oscillate with the following pattern:



We call this the "plus" polarization: it tidally stretches and squeezes along orthogonal axes that are aligned with the x and y axes of our coordinates. We denote this polarization h_+.

- Next, the opposite limit: $h^{\rm TT}_{xy}$ is a sinusoid, $h^{\rm TT}_{xx} = 0$. The separation vector now undergoes the following pattern as time goes on:



This pattern is the "cross" polarization, since it tidally stretches and squeezes along orthogonal axes aligned at 45° to our coordinates' x and y axes. We denote this polarization $h_\times$.

Recall that $h^{\rm TT}_{ij}$ has only two independent degrees of freedom: the 6 components of the symmetric tensor are constrained by 3 conditions to make it divergence free, plus 1 trace-free condition. The polarizations h_+ and $h_\times$ completely account for these two independent degrees of freedom.

16.4 Computing gravitational radiation from a source

We conclude this lecture by computing radiation given a source. To do this, we are going to take advantage of what we learned from our characterization of the gauge-invariant degrees of freedom in the linearized metric: we compute $h_{\alpha\beta}$ by some convenient method, and then from this project out the $h^{\rm TT}_{ij}$ pieces of this tensor. The result is guaranteed to be the gauge-invariant, physical radiation content of the spacetime that we construct.

A particularly convenient starting point is the solution we already wrote down using the radiative Green's function for the linearized Einstein equation in Lorenz gauge: the field equation

$$\Box \bar{h}_{\alpha\beta} = -16\pi G T_{\alpha\beta} \tag{16.20}$$

has the solution

$$\bar{h}_{\alpha\beta}(t, \mathbf{x}) = 4G \int d^3x' \frac{T_{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \;, \tag{16.21}$$

where $\mathbf{x}$ denotes a field point (the place where we measure $h_{\alpha\beta}$) and $\mathbf{x}'$ is the source point (a point within the source whose stress-energy tensor is $T_{\alpha\beta}$) that we integrate over.

When we evaluate a source of this kind, we are typically interested in a situation in which $|\mathbf{x} - \mathbf{x}'| \gg$ the size of the source. For example, in many astrophysical applications, important sources are tens to hundreds of kilometers in size, and the source may be tens of millions or tens of billions of *light years* from the field point. Noting that 1 light year is about 10^{13} kilometers, we see that the "$\gg$" written above if anything understates the extreme disparity between these two scales. We can thus treat $|\mathbf{x} - \mathbf{x}'|$ as simply r, the distance to the source, and our solution to very good approximation may be written

$$\bar{h}_{\alpha\beta} = \frac{4G}{r} \int d^3x' \, T_{\alpha\beta}(t - r; \mathbf{x}') \; . \tag{16.22}$$

A more careful expansion uses the fact that $1/|\mathbf{x} - \mathbf{x}'|$ can be written as a sum over Legendre polynomials, or over spherical harmonics. Inserting this expansion allows one to define a multipole expansion, and leads to higher-order corrections to the form that we are about to derive.

For us, only the spatial components are important, so we reduce (16.22) to

$$\bar{h}_{ij} = \frac{4G}{r} \int d^3x' \, T_{ij}(t - r; \mathbf{x}') \; . \tag{16.23}$$

We now take advantage of an Easter egg that was hidden very early on in his course: you showed the "tensor virial theorem" on problem set 2,

$$\int d^3\, x' \, T_{ij} = \frac{1}{2} \frac{d^2}{dt^2} \int d^3x' \, T_{00}(t - r; \mathbf{x}') x'_i x'_j \; . \tag{16.24}$$

Applying this to Eq. (16.23), we have

$$\begin{aligned} \bar{h}_{ij} &= \frac{2G}{r} \frac{d^2}{dt^2} \int d^3x' \, T_{00}(t - r; \mathbf{x}') x'_i x'_j \\ &= \frac{2G}{r} \frac{d^2 I_{ij}}{dt^2} \; . \end{aligned} \tag{16.25}$$

We now need to pull from this the transverse and traceless metric contribution.

Let's consider the transverse condition first. Far from the source, we have $\Box \bar{h}_{ij} = 0$, which suggests that we should expand our solution in plane waves:

$$\bar{h}_{jl} = A_{jl} e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \; . \tag{16.26}$$

The amplitude A_{jl} has no dependence on t or $\mathbf{x}$, though it may depend on ω or $\mathbf{k}$. A more complete presentation would write this as an integral over ω and $\mathbf{k}$; here we just examine things mode by mode[1], which is sufficient for our purposes.

Our "transverse" condition is that $\partial_j \bar{h}_{jl} = 0$. Let's enforce this:

$$\partial_j \bar{h}_{jl} = i k_j A_{jl}(\omega, \mathbf{k}) e^{i(\mathbf{k}\cdot x - \omega t)} = 0 \; . \tag{16.28}$$

This condition is equivalent to requiring that

$$k_j \bar{h}_{jl} = 0 \; . \tag{16.29}$$

[1]Such an expansion writes the waveform

$$\bar{h}_{jl} = \int d\omega \, d^3k \, A_{jl}(\omega, \mathbf{k}) e^{i(\mathbf{k}\cdot x - \omega t)} \; . \tag{16.27}$$

Everything given above carries over to this integral's integrand. Thanks to the linearity of this limit of general relativity, it is simple to combine each mode to assemble this integrated solution.

In other words, the wave has no components parallel to the direction of propagation. (This is why the form that I wrote down when we first began to explore radiative solutions had no z components — recall that it described a wave propagating along the z direction.) To guarantee that our solution satisfies this condition, define the projection tensor

$$P_{ij} = \delta_{ij} - n_i n_j \;, \qquad \text{where} \quad n_i = k_i / \sqrt{|\mathbf{k}|^2} \;. \tag{16.30}$$

Recalling some exercises from problem set 1 (problem 4 in particular), we recognize this as a tensor that projects tensor components into a subspace that is orthogonal to $\mathbf{n}$. The quantity $\mathbf{n}$ is in turn a unit vector (or unit 1-form) along the direction of propagation. It's also worth recalling from this homework exercise that the projection tensor P_{ij} can be regarded as the metric for geometric objects which live in that subspace orthogonal to $\mathbf{n}$. Using this, our transverse metric perturbation is

$$\bar{h}^{\rm T}_{ij} = \bar{h}_{lk} P_{li} P_{kj} \;. \tag{16.31}$$

Finally, we need to remove the trace. The general formula to make a general 2-index tensor[2] K_{ab} trace free is

$$K^{\rm TF}_{ab} = K_{ab} - \frac{1}{N}\left(g^{cd} K_{cd}\right) g_{ab} \;, \tag{16.32}$$

where g_{ab} is the metric describing the manifold in which the tensors K_{ab} and $K^{\rm TF}_{ab}$ reside, and N is the dimension of this manifold. (Note that $g^{ab} g_{ab} = N$.) For our current problem, the metric is P_{ij}, and $N = 2$. Applying this, and using the fact that raising and lowering indices is not meaningful in these spatial coordinates, we have

$$\bar{h}^{\rm TT}_{ij} = \bar{h}^{\rm T}_{ij} - \frac{1}{2} \bar{h}^{\rm T}_{mn} P_{mn} P_{ij} \;. \tag{16.33}$$

However, we also have

$$\bar{h}^{\rm T}_{mn} P_{mn} P_{ij} = \bar{h}_{lk} P_{lm} P_{kn} P_{mn} P_{ij} \;. \tag{16.34}$$

Using problem 4 on problem set 1 again,

$$P_{lm} P_{kn} P_{mn} P_{ik} = P_{lk} P_{ij} \;. \tag{16.35}$$

We thus finally have

$$\begin{aligned} \bar{h}^{\rm TT}_{ij} &= \bar{h}_{lk} P_{li} P_{kj} - \frac{1}{2} \bar{h}_{lk} P_{lk} P_{ij} \\ &= h^{\rm TT}_{ij} \;. \end{aligned} \tag{16.36}$$

In the final equality, we remove the overbar that denotes trace reversal. This is because once we've removed the trace, there's no distinction between the transverse-traceless metric perturbation $h^{\rm TT}_{ij}$ and its trace-reversed version $\bar{h}^{\rm TT}_{ij}$. Trace reversal has no effect when the trace is zero. (Though see the next section for a caveat when we apply this logic to realistic problems.) Using the solution we worked out earlier for $h^{\rm TT}_{ij}$, we have

$$\boxed{h^{\rm TT}_{ij} = \frac{2G}{r} \frac{d^2 I_{lk}}{dt^2} \left(P_{li} P_{kj} - \frac{1}{2} P_{lk} P_{ij}\right)} \tag{16.37}$$

The result is known as the *quadrupole formula* for the amplitude of gravitational radiation; it places a role in gravitational radiation theory quite similar to the role played by the dipole formula for the electromagnetic potential in electrodynamics.

[2]We use a and b for our indices because we are being agnostic about the nature of the geometry in which this tensor lives. It could be 4-dimensional spacetime; it could be N-dimensional space.

16.5 A subtlety regarding trace removal

A key step in the calculation above was equating $h_{ij}^{\rm TT}$ to $\bar{h}_{ij}^{\rm TT}$. This equality was justified by the idea that when the trace has been removed there is no difference between the two quantities. Our take-away slogan was that trace reversing has no effect when the trace is zero.

Strictly speaking, this is at best somewhat glib, and requires more careful justification. In particular, note that there are two notions of "trace" at play in this calculation, essentially because we are working with two different notions of "metric." We define $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - (h/2)\eta_{\alpha\beta}$, with $h \equiv \eta^{\alpha\beta} h_{\alpha\beta}$. The trace here is defined using the *spacetime* metric $\eta_{\alpha\beta}$. By contrast, we define the traceless nature of the field $h_{ij}^{\rm TT}$ by requiring that $\delta^{ij} h_{ij}^{\rm TT} = 0$. This field has zero trace with respect to the *spatial* metric δ_{ij}.

So if the "trace" has been removed can we really say that $\bar{h}_{ij}^{\rm TT} = h_{ij}^{\rm TT}$? If our spacetime consists of nothing but flat spacetime and gravitational radiation, then yes. However, if there are other sources of gravity that contribute to the gauge-invariant fields Φ and θ that we defined in Lecture 15, then the answer is no. Those fields affect the notion of trace that enters the definition of $\bar{h}_{\alpha\beta}$, and are completely absent from $h_{ij}^{\rm TT}$. Aspects of these two non-radiative gauge-invariant fields get mixed into $\bar{h}_{ij}^{\rm TT}$, so the equality asserted above cannot be strictly correct.

The reason why the equality nonetheless works very well is because of aspects of the theory of gravitational radiation that we will discuss in Lecture 17. A gravitational wave is strictly speaking only defined in a context in which we can split aspects of spacetime that vary on timescales and lengthscales, τ and λ, that are much shorter than the timescales and lengthscales, $\mathcal{T}$ and $\mathcal{L}$, on which the "background" varies. In this context, the fields Φ and Θ (which recall satisfy Poisson-time equations) will always vary only very slowly with time (if at all), and will only vary on very long lengthscales.

Physics is ultimately about measurement, and when we measure the properties of curved spacetime we do so over a set of lengthscales and timescales that are ultimately determined by our measuring apparatus. When we focus on the timescale and lengthscales that correspond to radiation, then we indeed discover that $\bar{h}_{ij}^{\rm TT} = h_{ij}^{\rm TT}$; the "contaminating" pieces of Φ and Θ that affect the trace $h = \eta^{\alpha\beta} h_{\alpha\beta}$ and thereby enter into $\bar{h}_{ij}$ do not affect the radiative aspects of our solution on the lengthscales and timescales that characterize gravitational radiation.

An application of this principle can be seen in how ground-based gravitational-wave antenna like LIGO and Virgo operate. The gravitational waves that they probe describe spacetime oscillations at frequencies of tens to thousands of Hertz. *No other* gravitational phenomenon near the Earth's surface varies on similar time scales, so even if the Poisson-type fields "contaminate" our notion of the trace-free metric term as we have constructed it above, those fields do not vary on timescales that these antennas can probe. The strongest time-varying non-radiative gravitational term which the detectors have to worry about is the contribution to spacetime from the Moon, which varies on much longer timescales due to the Moon's orbit and the rotation of the Earth. (Of course, many *non*-gravitational phenomena oscillate in the antenna's sensitive band, generating noise. As such, it is hugely important to shield the test masses which probe for gravitational radiation from these noise sources. As detectors get more sensitive, local gravity may start to become a source of noise: future detectors are expected to be limited at the lowest frequencies by gravitational coupling to seismic modes in the ground on which the detectors are built. See https://arxiv.org/abs/gr-qc/9806018 for detailed discussion.)