Massachusetts Institute of Technology
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Lecture 17
Gravitational radiation on a curved background; radiative backreaction

17.1 Gravitational radiation in a general background spacetime: Conceptual foundation

Our detailed discussion of gravitational waves so far defines these waves as "small" perturbations about a flat background spacetime. A lot of useful results and formalism grow out of this analysis, but this definition is quite restrictive. In the general case, we imagine that waves are propagating on some reasonably well characterized but *non*-flat background:

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + h_{\alpha\beta}\ . \tag{17.1}$$

The background $\hat{g}_{\alpha\beta}$ can now be something that varies in space and time; the perturbation $h_{\alpha\beta}$ will typically be small, though there may be circumstances where it is not as small as we imagined in our previous discussion.

Within this framework, how do we even define what part of the spacetime is "wave" and what part is "background"? Local measurements only measure some surrogate of the total spacetime metric $g_{\alpha\beta}$. Based on what we have laid out so far, a measurement does not enable use to clearly distinguish what is background from what is radiation. How do we distinguish between the radiative aspects of spacetime and its non-radiative, background aspects?

The trick to separating "radiation" from "background" is to use the idea that the radiation and background vary on very different scales. We imagine that the radiation varies on timescales τ and lengthscales λ that are far shorter than the timescales $\mathcal{T}$ and lengthscales $\mathcal{L}$ which describe the background. Derivatives associated with radiation will thus introduce scalings $\sim 1/\tau$ and $\sim 1/\lambda$ into the analysis[1]; derivatives associated with the background introduce scalings $\sim 1/\mathcal{T}$ and $\sim 1/\mathcal{L}$. A useful analogy is to think of water waves propagating on an ocean. If one imagines oneself to be an ant riding a cork that floats on the water's surface, you just follow the water's behavior, and are agnostic about whether you are in a wave, are following the curvature of the Earth, or are being affected by some local structure in the water. But if we zoom out and examine this cork from a larger perspective, we can use a separation of scales to tell whether the cork is rising and falling due to a passing wave, or is being affected by the curvature of the Earth or a local structure. Waves are associated with small scale, rapid oscillations; aspects of the dynamics related to larger-scale, long-duration structures are not impacted in such a small scale, rapid way. Our goal now is to understand how to similarly separate scales in our discussion of the dynamics of spacetime.

17.2 Gravitational radiation in a general background spacetime: Technical details

We now develop the concepts laid out above. We assume that the background varies on long scales $(\mathcal{T}, \mathcal{L})$. To set some context, it is useful to put numbers to these quantities: in a frame attached to the surface of the earth, $\mathcal{L}$ is on the order of $\sqrt{R^3_{\rm Earth}/GM_{\rm Earth}} \simeq 2.4 \times 10^8\,\mathrm{km}$, the typical

[1] In units with $c = 1$, $\tau = \lambda$. We maintain a distinction because there is no reason that the long scales are connected in a similar way, so we will often want to separately compare timescales and lengthscales.

length scale[2] associated with the Earth's spacetime curvature. In this frame, $\mathcal{T}$ is on the order of several hours, the timescale over which the Earth's rotation changes the orientation of this frame with respect to the solar system (thus changing the magnitude of tidal effects from the Moon and the Sun). Any gravitational wave will need to vary on short timescales (τ, λ) that are much smaller than $(\mathcal{T}, \mathcal{L})$.

We will implement this idea by defining an *averaging* operation: Imagine there exists a scale $L \sim$ (several)λ, $T \sim$ (several)τ, where "several" is small enough that $L \ll \mathcal{L}$, $T \ll \mathcal{T}$. Our background is then given by averaging spacetime over this scale, and the wave by subtracting this average from the total:

$$\hat{g}_{\alpha\beta} \equiv \langle g_{\alpha\beta} \rangle \;, \qquad h_{\alpha\beta} = g_{\alpha\beta} - \hat{g}_{\alpha\beta} \;. \tag{17.2}$$

A version of this averaging procedure was introduced in a paper by Dieter Brill and James Hartle[3], and adapted to the form that we use here by R. A. Isaacson[4]. The averaging consists of integrating the metric (or any other tensor) against a weighting function as follows:

$$\langle g_{\alpha\beta}(x^\mu) \rangle = \int d^4x' \, g_{\alpha'\beta'}[(x')^\mu] \, f^{\alpha'\beta'}_{\alpha\beta}[x^\mu, (x')^\mu] \;. \tag{17.3}$$

Note that when we implement this procedure, we are necessarily taking a somewhat coarse-grained view of the nature of spacetime, or at least of the background spacetime. Again, this is reminiscent of the water-wave metaphor: we can only sensibly define the action of the wave if we move away from viewing the water in the immediate vicinity of the floating cork and cast our view out on a scale similar to the wavelength of the waves themselves.

The weighting function $f^{\alpha'\beta'}_{\alpha\beta}[x^\mu, (x')^\mu]$ we have introduced here allows one to transport tensorial quantities from an event $(x')^\mu$ to an event x^μ. It is taken to be peaked when $x^\mu = (x')^\mu$, it has a width of order L, T, and is normalized such that

$$\int d^4x' \, f^{\alpha'\beta'}_{\alpha\beta}[x^\mu, (x')^\mu] = \delta^{\alpha'}{}_\alpha \delta^{\beta'}{}_\beta \;. \tag{17.4}$$

As shown in detail by Isaacson, this averaging procedure "washes out" oscillations on a scale $\lambda < L$, $\tau < T$, but does not have a large impact on features which vary on the long scales $\mathcal{L}, \mathcal{T}$. The output of this procedure is in fact a tensor up to errors on the order of the ratio of scales. In other words, when one changes representation, one finds that $\hat{g}_{\alpha\beta}$ follows the tensor transformation rule up to errors $\mathcal{O}[(\tau/\mathcal{T}), (\lambda/\mathcal{L})]$.

17.2.1 Christoffel symbols and curvature tensors for perturbation to a curved background

With this in mind, let's now develop the mathematical machinery which describes gravitational radiation in this framework. Writing $g_{\alpha\beta} = \hat{g}_{\alpha\beta} + h_{\alpha\beta}$, we will develop the connection and curvature tensors, expanding all quantities to linear order in the wave amplitude h. We begin by noting that

$$g^{\alpha\beta} = \hat{g}^{\alpha\beta} - h^{\alpha\beta} \;, \qquad \text{where} \quad h^{\alpha\beta} = \hat{g}^{\alpha\mu}\hat{g}^{\beta\nu}h_{\mu\nu} \;. \tag{17.5}$$

[2]We use the idea that the components of the Riemann tensor in a Cartesian-like coordinate representation have dimensions $1/(\text{length})^2$, and have defined this length as $\sqrt{1/(\text{typical Riemann component})}$.

[3]D. B. Brill and J. B. Hartle, Phys. Rev. **135**, 271 (1964); a link to this paper will be posted to the 8.962 webpage.

[4]R. A. Isaacson, Phys. Rev. **166**, 1272 (1968); a link to this paper will be posted to the 8.962 webpage.

It is simple to verify this by evaluating $g^{\alpha\mu}g_{\mu\beta}$ and discarding terms of $\mathcal{O}(h^2)$. Using this, let's assemble the Christoffel connection:

$$\begin{aligned}
\Gamma^\alpha{}_{\mu\nu} &= \frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\nu\beta} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right) \\
&= \frac{1}{2}\left(\hat{g}^{\alpha\beta} - h^{\alpha\beta}\right)\left(\partial_\mu \hat{g}_{\nu\beta} + \partial_\nu \hat{g}_{\beta\mu} - \partial_\beta \hat{g}_{\mu\nu} - \partial_\mu h_{\nu\beta} - \partial_\nu h_{\beta\mu} + \partial_\beta h_{\mu\nu}\right) \\
&= \hat{\Gamma}^\alpha{}_{\mu\nu} - h^{\alpha\beta}\hat{g}_{\beta\gamma}\hat{\Gamma}^\gamma{}_{\mu\nu} + \frac{1}{2}\hat{g}^{\alpha\beta}\left(\partial_\mu h_{\nu\beta} + \partial_\nu h_{\beta\mu} - \partial_\beta h_{\mu\nu}\right) + \mathcal{O}(h^2)\ . \qquad (17.6)
\end{aligned}$$

Here, $\hat{\Gamma}^\alpha{}_{\mu\nu}$ means a Christoffel connection defined in the background spacetime. Let us define $\hat{\nabla}_\mu$ as a covariant derivative taken with respect to the background. We can replace the partial derivatives in the above expression with $\hat{\nabla}_\mu$ at the cost of introducing additional background Christoffels coupling to the perturbation. Doing so and organizing terms, we find that the result can be written

$$\Gamma^\alpha{}_{\mu\nu} = \hat{\Gamma}^\alpha{}_{\mu\nu} + \delta\Gamma^\alpha{}_{\mu\nu}\ , \qquad (17.7)$$

where

$$\delta\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2}\hat{g}^{\alpha\beta}\left(\hat{\nabla}_\mu h_{\nu\beta} + \hat{\nabla}_\nu h_{\beta\mu} - \hat{\nabla}_\beta h_{\mu\nu}\right)\ . \qquad (17.8)$$

Continuing in this fashion, it is straightforward to show that the Riemann tensor takes the form

$$R^\alpha{}_{\beta\gamma\delta} = \hat{R}^\alpha{}_{\beta\gamma\delta} + \delta R^\alpha{}_{\beta\gamma\delta}\ , \qquad (17.9)$$

where $\hat{R}^\alpha{}_{\beta\gamma\delta}$ is assembled from $\hat{g}_{\alpha\beta}$ in the usual way, but where we find

$$\delta R^\alpha{}_{\beta\gamma\delta} = \hat{\nabla}_\gamma \delta\Gamma^\alpha{}_{\beta\delta} - \hat{\nabla}_\delta \delta\Gamma^\alpha{}_{\beta\gamma}\ . \qquad (17.10)$$

17.2.2 Generalizing gauge transformations to a curved background

Before moving on, an important tool for what is to come is the notion of an infinitesimal coordinate transformation on a curved background, which will generalize the notion of a gauge transformation to such a background. We begin with a spacetime whose metric we represent as some background $\hat{g}_{\alpha\beta}$ plus a perturbation $h_{\alpha\beta}$; these are all expressed in terms of the coordinate x^μ. We wish to change to a new coordinate y^μ which is related to x^μ by an infinitesimal shift ξ^μ:

$$y^\mu = x^\mu + \xi^\mu \quad \rightarrow \quad x^\mu = y^\mu - \xi^\mu(y^\gamma)\ . \qquad (17.11)$$

(This second form is a somewhat more convenient form for the analysis we need to perform.) This coordinate transformation changes the metric according to

$$g_{\alpha\beta}(y^\mu) = g_{\gamma\delta}(x^\mu)\frac{\partial x^\gamma}{\partial y^\alpha}\frac{\partial x^\delta}{\partial y^\beta}\ . \qquad (17.12)$$

Expand this expression:

$$\begin{aligned}
g_{\alpha\beta}(y^\mu) &= \left(\hat{g}_{\gamma\delta} + h_{\gamma\delta}\right)\left(\delta^\gamma{}_\alpha - \partial_\alpha\xi^\gamma\right)\left(\delta^\delta{}_\beta - \partial_\beta\xi^\delta\right) \\
&= \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{g}_{\gamma\beta}\partial_\alpha\xi^\delta - \hat{g}_{\alpha\delta}\partial_\beta\xi^\gamma + \mathcal{O}[h\,\partial\xi, (\partial\xi)^2] \\
&= \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{g}_{\gamma\beta}(\hat{\nabla}_\alpha\xi^\gamma - \xi^\delta\hat{\Gamma}^\gamma{}_{\alpha\delta}) - \hat{g}_{\alpha\delta}(\hat{\nabla}_\beta\xi^\delta - \xi^\gamma\hat{\Gamma}^\delta{}_{\beta\gamma}) \\
&= \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{\nabla}_\alpha\xi_\beta - \hat{\nabla}_\beta\xi_\alpha + \xi^\gamma(\hat{\Gamma}_{\beta\alpha\gamma} + \hat{\Gamma}_{\alpha\beta\gamma}) \\
&= \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{\nabla}_\alpha\xi_\beta - \hat{\nabla}_\beta\xi_\alpha + \xi^\gamma\partial_\gamma\hat{g}_{\alpha\beta}\ . \qquad (17.13)
\end{aligned}$$

On going from the fourth to the fifth line, we've used the definition of the Christoffel symbol. On that last line, the background in this expression is currently expressed as a function of the old coordinates, x^γ. We want it to appear as a function of the new coordinates, y^μ:

$$\begin{aligned}\hat{g}_{\alpha\beta}(x^\mu) &= \hat{g}_{\alpha\beta}(y^\mu - \xi^\mu)\\ &= \hat{g}_{\alpha\beta}(y^\mu) - \xi^\mu\partial_\mu\hat{g}_{\alpha\beta}\ .\end{aligned} \tag{17.14}$$

Note, we could do this expansion for the perturbation as well: $h_{\alpha\beta}(x^\mu) = h_{\alpha\beta}(y^\mu - \xi^\mu) = h_{\alpha\beta}(y^\mu) - \xi^\mu\partial_\mu h_{\alpha\beta}$. This means that $h_{\alpha\beta}(x^\mu) = h_{\alpha\beta}(y^\mu)$ plus terms that are of order "small squared." We neglect these "small squared" contributions.

Substituting the result for $\hat{g}_{\alpha\beta}(x^\mu)$ into $g_{\alpha\beta}(y^\mu)$ yields

$$g_{\alpha\beta}(y^\mu) = \hat{g}_{\alpha\beta} + h_{\alpha\beta} - \hat{\nabla}_\alpha\xi_\beta - \hat{\nabla}_\beta\xi_\alpha\ , \tag{17.15}$$

from which we read off

$$h_{\alpha\beta} \to h_{\alpha\beta} - \hat{\nabla}_\alpha\xi_\beta - \hat{\nabla}_\beta\xi_\alpha\ . \tag{17.16}$$

Equation (17.16) generalizes the notion of a gauge transformation to a curved background.

17.2.3 The Einstein tensor for a perturbation to a curved background

We now are at last ready to assemble the Einstein tensor. For simplicity, we will do this in vacuum; this will give us an equation that describes how radiation propagates on a curved background. Making this simplification allows us to set $\hat{R}_{\mu\nu} = \hat{G}_{\mu\nu} = 0$; it is straightforward but tedious to lift this approximation, and the resulting equations are significantly more complicated to work with. As you will see, what results from this already simplified analysis is complicated enough for now.

The piece we need to start this analysis comes from the Riemann term $\delta R^\alpha{}_{\mu\beta\nu}$. Writing this out in terms of the perturbation, we find that

$$\delta R^\alpha{}_{\mu\beta\nu} = \frac{1}{2}\left(\hat{\nabla}_\beta\hat{\nabla}_\mu h^\alpha{}_\nu + \hat{\nabla}_\beta\hat{\nabla}_\nu h^\alpha{}_\mu - \hat{\nabla}_\beta\hat{\nabla}^\alpha h_{\mu\nu} - \hat{\nabla}_\nu\hat{\nabla}_\mu h^\alpha{}_\beta - \hat{\nabla}_\nu\hat{\nabla}_\beta h^\alpha{}_\mu + \hat{\nabla}_\nu\hat{\nabla}^\alpha h_{\mu\beta}\right)\ . \tag{17.17}$$

Contracting on α and β yields

$$\delta R_{\mu\nu} = \frac{1}{2}\left(\hat{\nabla}_\alpha\hat{\nabla}_\mu h^\alpha{}_\nu + \hat{\nabla}_\alpha\hat{\nabla}_\nu h^\alpha{}_\mu - \hat{\Box} h_{\mu\nu} - \hat{\nabla}_\nu\hat{\nabla}_\mu h\right)\ . \tag{17.18}$$

Note that the final two terms inside parentheses on the right-hand side of (17.17) sum to zero when we contract on α and β. We have introduced $\hat{\Box} = \hat{g}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu$, a covariant wave operator on the background spacetime. The Ricci scalar corresponding to this is given by

$$\delta R = \hat{g}^{\mu\nu}R_{\mu\nu} = -\hat{\Box}h + \hat{\nabla}_\alpha\hat{\nabla}_\beta h^{\beta\alpha}\ . \tag{17.19}$$

With all this in hand, we finally assemble the Einstein tensor[5]:

$$\begin{aligned}\delta G_{\mu\nu} &= \delta R_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\delta R\\ &= -\frac{1}{2}\hat{\Box}h_{\mu\nu} + \frac{1}{2}\hat{g}_{\mu\nu}\hat{\Box}h - \frac{1}{2}\hat{\nabla}_\nu\hat{\nabla}_\mu h + \frac{1}{2}\hat{\nabla}_\alpha\hat{\nabla}_\mu h^\alpha{}_\nu + \frac{1}{2}\hat{\nabla}_\alpha\hat{\nabla}_\nu h^\alpha{}_\mu - \frac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\alpha\hat{\nabla}_\beta h^{\beta\alpha}\ .\end{aligned} \tag{17.20}$$

[5]Note that if we were not in a vacuum, $\delta G_{\mu\nu}$ would contain an additional term proportional to $h_{\mu\nu}\hat{R}$. This is an example of how this problem becomes more complicated as we go beyond this simple first exposition of these ideas.

Just as when we first assembled the analog of this tensor with a flat background, the result is somewhat unwieldy. We can clean it up significantly by replacing the metric perturbation with its trace-reversed version: we put

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h\, \hat{g}_{\mu\nu} , \qquad \text{where} \quad h \equiv \hat{g}^{\alpha\beta} h_{\alpha\beta} . \tag{17.21}$$

Using this, we substitute $h_{\mu\nu} = \bar{h}_{\mu\nu} + (h/2)\hat{g}_{\mu\nu}$ and use the fact that $\hat{\nabla}_\alpha \hat{g}_{\mu\nu} = 0$. The result is

$$\delta G_{\mu\nu} = -\frac{1}{2}\hat{\Box}\bar{h}_{\mu\nu} + \frac{1}{2}\hat{\nabla}_\alpha \hat{\nabla}_\mu \bar{h}^\alpha{}_\nu + \frac{1}{2}\hat{\nabla}_\alpha \hat{\nabla}_\nu \bar{h}^\alpha{}_\mu - \frac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{h}^{\beta\alpha} . \tag{17.22}$$

This is similar in form to what we found when we developed this tensor on a flat background. However, in that case, all of the derivatives which appeared after the wave operator were partial derivatives. We were able to rearrange all of them so that they took the form of a divergence of the metric perturbation. We then selected a gauge such that the divergence vanished, and found that the wave had a much simpler form.

Our derivatives are now covariant rather than partial, so when we exchange their order, we find a coupling to the background Riemann tensor. Let's examine this for the two terms in (17.22) that are the most problematic:

$$\begin{aligned} \hat{\nabla}_\alpha \hat{\nabla}_\mu \bar{h}^\alpha{}_\nu &= \hat{\nabla}_\mu \hat{\nabla}_\alpha \bar{h}^\alpha{}_\nu + \hat{R}^\alpha{}_{\beta\alpha\mu}\bar{h}^\beta{}_\nu - \hat{R}^\beta{}_{\nu\alpha\mu}\bar{h}^\alpha{}_\beta \\ &= \hat{\nabla}_\mu \hat{\nabla}_\alpha \bar{h}^\alpha{}_\nu - \hat{R}_{\beta\nu\alpha\mu}\bar{h}^{\alpha\beta} \\ &= \hat{\nabla}_\mu \hat{\nabla}_\alpha \bar{h}^\alpha{}_\nu - \hat{R}_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} . \end{aligned} \tag{17.23}$$

We used the fact that $\hat{R}^\alpha{}_{\beta\alpha\nu} = \hat{R}_{\beta\nu}$, which is zero by our assumption that the spacetime is vacuum. We also used a Riemann symmetry to arrange the indices on the final line. Repeating this exercise for the other term of this form yields

$$\hat{\nabla}_\alpha \hat{\nabla}_\nu \bar{h}^\alpha{}_\mu = \hat{\nabla}_\nu \hat{\nabla}_\alpha \bar{h}^\alpha{}_\mu - \hat{R}_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} . \tag{17.24}$$

To get this form, we used Riemann symmetry, the symmetry of the metric, and relabeled dummy indices. Our expression for the Einstein tensor now becomes

$$\delta G_{\mu\nu} = -\frac{1}{2}\hat{\Box}\bar{h}_{\mu\nu} - \hat{R}_{\alpha\mu\beta\nu}\bar{h}^{\alpha\beta} + \frac{1}{2}\hat{\nabla}_\mu \hat{\nabla}_\alpha \bar{h}^\alpha{}_\nu + \frac{1}{2}\hat{\nabla}_\nu \hat{\nabla}_\alpha \bar{h}^\alpha{}_\nu - \frac{1}{2}\hat{g}_{\mu\nu}\hat{\nabla}_\alpha \hat{\nabla}_\beta \bar{h}^{\beta\alpha} . \tag{17.25}$$

As we justify in the next subsection, we can choose our gauge such that $\hat{\nabla}_\alpha \bar{h}^\alpha{}_\nu = 0$; we can further set gauge such that the metric perturbation is trace free. This choice is known as "TT" gauge. In TT gauge, there is no difference between $\bar{h}_{\mu\nu}$ and $h_{\mu\nu}$, and our perturbed Einstein equation yields the following equation for the propagation of gravitational waves on a curved background:

$$\boxed{\boxed{\hat{\Box}h_{\mu\nu} + 2\hat{R}_{\alpha\mu\beta\nu}h^{\alpha\beta} = 0}} \tag{17.26}$$

Equation (17.26) describes how gravitational waves propagate over a curved background. In addition to having dynamics determined by the wave operator $\hat{\Box}$, we see that radiation "scatters" off of spacetime curvature. (Note that the presentation of this equation in Flanagan and Hughes 2005 has a minus sign error in front of the Riemann term.)

17.2.4 Lorenz gauge and TT gauge

To get our final wave equation, we asserted that we could put the metric perturbation into a gauge that is divergence free. Let's examine this condition more closely. Under a change of gauge, we change from $h^{\rm old}_{\mu\nu}$ to

$$h^{\rm new}_{\mu\nu} = h^{\rm old}_{\mu\nu} - \hat{\nabla}_\mu \xi_\nu - \hat{\nabla}_\nu \xi_\mu . \tag{17.27}$$

Applying this to the trace-reversed perturbation, we find

$$\bar{h}^{\rm new}_{\mu\nu} = \bar{h}^{\rm old}_{\mu\nu} - \hat{\nabla}_\mu \xi_\nu - \hat{\nabla}_\nu \xi_\mu + \hat{g}_{\mu\nu} \hat{\nabla}^\alpha \xi_\alpha \; . \tag{17.28}$$

Let's take the covariant background divergence of this:

$$\hat{\nabla}^\mu \bar{h}^{\rm new}_{\mu\nu} = \hat{\nabla}^\mu \bar{h}^{\rm old}_{\mu\nu} - \hat{\Box}\xi_\nu - \hat{\nabla}^\mu \hat{\nabla}_\nu \xi_\mu + \hat{\nabla}_\nu \nabla^\alpha \xi_\alpha \; . \tag{17.29}$$

Let's focus on that final term, $\hat{\nabla}_\nu \nabla^\alpha \xi_\alpha = \hat{\nabla}_\nu \nabla^\mu \xi_\mu = \hat{\nabla}_\nu \hat{\nabla}_\mu \xi^\mu$ (relabeling the dummy index). If we could exchange the order of derivatives, this would cancel perfectly with the preceding term, leaving us with a fairly simple condition. However, doing such an exchange introduces a factor of the Riemann tensor:

$$\begin{aligned}
\hat{\nabla}_\nu \hat{\nabla}_\mu \xi^\mu &= \hat{\nabla}_\mu \hat{\nabla}_\nu \xi^\mu + \hat{R}^\mu{}_{\gamma\nu\mu} \xi^\gamma \\
&= \hat{\nabla}_\mu \hat{\nabla}_\nu \xi^\mu - \hat{R}_{\gamma\nu} \xi^\gamma \\
&= \hat{\nabla}_\mu \hat{\nabla}_\nu \xi^\mu \\
&= \hat{\nabla}^\mu \hat{\nabla}_\nu \xi_\mu \; .
\end{aligned} \tag{17.30}$$

In going from the first to the second lines, we use the fact that the Riemann tensor which enters this expression is contracted on indices 1 and 4, yielding the (negative) Ricci tensor. But Ricci is zero by our simplifying assumption that the background is a vacuum solution. So, in fact we *can* exchange those derivatives, and we find

$$\hat{\nabla}^\mu \bar{h}^{\rm new}_{\mu\nu} = \hat{\nabla}^\mu \bar{h}^{\rm old}_{\mu\nu} - \hat{\Box}\xi_\nu \; . \tag{17.31}$$

We can thus develop a curved spacetime Lorenz gauge, allowing us to set all divergences of $\bar{h}_{\mu\nu}$ to zero by choosing gauge generators such that

$$\hat{\Box}\xi_\nu = \hat{\nabla}\bar{h}^{\rm old}_{\mu\nu} \; . \tag{17.32}$$

Solutions of this exist everywhere that the background satisfies the vacuum Einstein equations. If the background were not vacuum, this condition would be modified by a coupling of the gauge generator to the background Ricci tensor, which implies a coupling the stress-energy tensor; we comment on such situations in the next subsection.

There isn't just one Lorenz gauge: if ξ_ν is the generator of Lorenz gauge, and χ_ν satisfies $\hat{\Box}\chi_\nu = 0$, then $\xi_\nu + \chi_\nu$ is also a generator of Lorenz gauge. Let us assume that $\bar{h}^{\rm L}_{\mu\nu}$ is a trace-reversed metric perturbation that satisfies Lorenz gauge. Let's change gauge once again using a generator χ_ν that satisfies $\hat{\Box}\chi_\nu = 0$:

$$\bar{h}^{\rm new}_{\mu\nu} = \bar{h}^{\rm L}_{\mu\nu} - \hat{\nabla}_\mu \chi_\nu - \hat{\nabla}_\nu \chi_\mu + \hat{g}_{\mu\nu} \hat{\nabla}^\alpha \chi_\alpha \; . \tag{17.33}$$

Let's take the trace of this:

$$\bar{h}^{\rm new} = \bar{h}^{\rm L} - 2\hat{\nabla}^\nu \chi_\nu + 4\hat{\nabla}^\alpha \chi_\alpha = \hat{h}^{\rm L} + 2\hat{\nabla}^\nu \chi_\nu \; . \tag{17.34}$$

This means that if we choose these generators such that

$$\hat{\nabla}^\nu \chi_\nu = -\frac{1}{2}\bar{h}^{\rm L} \; , \tag{17.35}$$

then the trace of the new metric is zero. Choosing this gauge means that there is no difference between $\bar{h}_{\mu\nu}$ and $h_{\mu\nu}$. This defines the TT gauge.

17.2.5 Considerations on a non-vacuum background

At several points in the above discussion, we used the fact that $T_{\mu\nu} = 0$ and thus $R_{\mu\nu} = 0$ in order to simplify the equations. By inspection it is clear that if we could not set these terms to zero then the discussion would become at a minimum more complicated. How do we handle this situation?

The key is to use the separation of scales again. Regions which are non vacuum vary on long length and time scales; the radiation varies on much shorter scales. By taking the Fourier transform of relevant quantities and examining how they behave in frequency and wavenumber space, one can develop useful relationships which are similar to those we have developed here, though with some modifications. As long as the separation of scales is valid, these modifications are not too difficult to develop and work with.

17.3 Energy content of gravitational waves: The Isaacson stress-energy tensor

We now have a good prescription for generating gravitational waves, and for studying how they evolve as they propagate out across spacetime, nicely analogous to our framework for studying the generation and propagation of electromagnetic radiation. One should recall that an important aspect of electromagnetic radiation is that it backreacts on a source, carrying away energy. Does gravitational radiation likewise carry away energy? This was quite a hotly debated topic in the mid 20th century (with some holdouts continuing to debate the issue into the late 1980s). The fact that general relativity can be a complicated nonlinear mess makes deducing the answer not exactly straightforward. On the one hand, one could argue that in a freely falling, locally Lorentzian representation, spacetime always just looks like special relativity. There's nothing there in this frame, so how can it carry energy?

Richard Feynman put forward one of the first clear physical arguments[6] that this radiation must carry energy: Imagine a very long stick on which there are two rings, free to slide up and down the stick, but with a small amount of friction. Place those rings on opposite ends of the stick, and allow the whole apparatus to float in free-fall (say in a high orbit about the Earth). The center of mass of the stick follows a geodesic; the rest of the stick follows along since the stick's molecules are bound to that center of mass. The two rings follow geodesics of their own, modulo a slight frictional coupling to the stick. Now, a gravitational wave comes by. It has no significant impact on the stick; the intermolecular forces within the stick effectively oppose the squeezing and stretching that the wave exerts. But by the equation of geodesic deviation, the rings slide back and forth, responding to the waves as they float in free fall. They rub a bit against the stick, and because of the slight frictional coupling generate a little bit of heat. The gravitational wave has thus heated up the stick; this wave must therefore carry energy.

The concerns about a freely-falling frame are not misplaced, however; note that a key part of Feynman's argument is that we need an extended object so that the impact of a gravitational wave can be discerned. Indeed, key to making a notion of energy carried by these waves work is that we will need to examine important quantities over some finite-sized region of spacetime. We also will need to go to second order in perturbation theory, since all first-order terms vanish when we average over such a region. This analysis was first done by Isaacson; our discussion is a synopsis of what Isaacson developed. We begin by writing the metric of spacetime

$$g_{\alpha\beta} = \hat{g}_{\alpha\beta} + \varepsilon h_{\alpha\beta} + \varepsilon^2 j_{\alpha\beta} \,. \qquad (17.36)$$

We have here introduced an order-counting parameter ε. The value of this parameter is 1; however, its powers allow us to keep track of the order in perturbation theory that we are studying.

[6]This is discussed in Daniel Kennefick's book *Traveling at the Speed of Thought* (Princeton University Press, 2007; see pages 134-136). This book goes through the travails of the history of gravitational waves, and contains a lot of fascinating and occasionally sordid details.

Let us focus on the vacuum Einstein equation, setting $T_{\alpha\beta} = 0$. We write

$$\begin{aligned} 0 &= G_{\alpha\beta}\left[\hat{g}_{\mu\nu} + \varepsilon h_{\mu\nu} + \varepsilon^2 j_{\mu\nu}\right] \\ &= G^{(0)}_{\alpha\beta}\left[\hat{g}_{\mu\nu}\right] + \varepsilon G^{(1)}_{\alpha\beta}\left[h_{\mu\nu};\hat{g}_{\mu\nu}\right] + \varepsilon^2 G^{(1)}_{\alpha\beta}\left[j_{\mu\nu};\hat{g}_{\mu\nu}\right] + \varepsilon^2 G^{(2)}_{\alpha\beta}\left[h_{\mu\nu};\hat{g}_{\mu\nu}\right] \ . \end{aligned} \tag{17.37}$$

The meaning of these terms is as follows:

$$\begin{aligned} G^{(0)}_{\alpha\beta}\left[\hat{g}_{\mu\nu}\right] &\quad\rightarrow\quad \text{"Standard" Einstein tensor} \\ G^{(1)}_{\alpha\beta}\left[h_{\mu\nu};\hat{g}_{\mu\nu}\right] &\quad\rightarrow\quad \text{1st-order correction} \\ G^{(1)}_{\alpha\beta}\left[j_{\mu\nu};\hat{g}_{\mu\nu}\right] &\quad\rightarrow\quad \text{Operator for 1st-order correction acting on the 2nd-order piece of metric} \\ G^{(2)}_{\alpha\beta}\left[h_{\mu\nu};\hat{g}_{\mu\nu}\right] &\quad\rightarrow\quad \text{Messy 2nd-order correction to the Einstein tensor.} \end{aligned}$$

The last piece, $G^{(2)}_{\alpha\beta}[h_{\mu\nu};\hat{g}_{\mu\nu}]$, introduces many terms of the schematic form (suppressing indices) $h\hat{\nabla}\hat{\nabla}h$ or $(\hat{\nabla}h)(\hat{\nabla}h)$. Details can be found in Isaacson's paper; I will write out the key terms at a relevant point below.

Key for us is that we require the Einstein field equation to hold at each order in ε:

- $\mathcal{O}(1)$: $G^{(0)}_{\alpha\beta}[\hat{g}_{\mu\nu}] = 0$. This is a statement that $\hat{g}_{\mu\nu}$ is a vacuum solution to the field equation.
- $\mathcal{O}(\varepsilon)$: $G^{(1)}_{\alpha\beta}[h_{\mu\nu};\hat{g}_{\mu\nu}] = 0$: This reproduces the analysis we did leading to the wave equation $\hat{\Box}h_{\mu\nu} + 2R_{\alpha\mu\beta\nu}h^{\alpha\beta} = 0$.
- $\mathcal{O}(\varepsilon^2)$: $G^{(1)}_{\alpha\beta}[j_{\mu\nu};\hat{g}_{\mu\nu}] = -G^{(2)}_{\alpha\beta}[h_{\mu\nu};\hat{g}_{\mu\nu}]$. The 2nd order perturbation obeys a *sourced* wave equation, with terms of order h^2 acting as this source.

In other words, terms that are quadratic in the wave amplitude act as a source for the 2nd-order corrections to spacetime.

Let us use the notion of averaging to separate the 2nd-order contributions as follows:

$$j_{\mu\nu} = \langle j_{\mu\nu}\rangle + \Delta j_{\mu\nu} \ . \tag{17.38}$$

The averaged term $\langle j_{\mu\nu}\rangle$ varies on the long scales $(\mathcal{T},\mathcal{L})$; the oscillation $\Delta j_{\mu\nu}$ varies on the short scales (τ,λ). This suggests that we regroup terms in the metric according to how they vary on different scales:

$$g_{\mu\nu} = \left[\hat{g}_{\mu\nu} + \varepsilon^2\langle j_{\mu\nu}\rangle\right]_{\rm L} + \left[\varepsilon h_{\mu\nu} + \varepsilon^2\Delta j_{\mu\nu}\right]_{\rm S} \ . \tag{17.39}$$

The subscripts "L" and "S" denote terms that vary on the long and short scales, respectively. Let's apply the Brill-Hartle averaging to the $\mathcal{O}(\varepsilon^2)$ Einstein tensor; doing so yields

$$\left\langle G^{(1)}_{\alpha\beta}\left[j_{\mu\nu};\hat{g}_{\mu\nu}\right]\right\rangle = -\left\langle G^{(2)}_{\alpha\beta}\left[h_{\mu\nu};\hat{g}_{\mu\nu}\right]\right\rangle \ . \tag{17.40}$$

Here we can use a trick make possible by the Brill-Hartle averaging procedure: when we average the second derivative of a tensor function, the result is very close to the second derivative of the average of the function:

$$\left\langle\partial^2 f_{\mu\nu}\right\rangle = \partial^2\left\langle f_{\mu\nu}\right\rangle + \mathcal{O}\left[(\tau/\mathcal{T})^2,(\lambda/\mathcal{L})^2\right] \ . \tag{17.41}$$

Using this, our averaged Einstein equation becomes

$$G^{(1)}_{\alpha\beta}\left[\langle j_{\mu\nu}\rangle\,;\hat{g}_{\mu\nu}\right] = -\left\langle G^{(2)}_{\alpha\beta}\left[h_{\mu\nu};\hat{g}_{\mu\nu}\right]\right\rangle \ . \tag{17.42}$$

Because the form on the left-hand side is linear in $\langle j_{\mu\nu}\rangle$, we can modify it further, writing

$$G_{\alpha\beta}\left[\hat{g}_{\mu\nu}+\varepsilon^2\langle j_{\mu\nu}\rangle\right] = -\left\langle G^{(2)}_{\alpha\beta}\left[h_{\mu\nu};\hat{g}_{\mu\nu}\right]\right\rangle \ . \tag{17.43}$$

This is a *very* interesting representation of the Einstein field equation. It is saying that the Einstein tensor for the long-lengthscale varying metric terms is governed by a source that looks like an averaged 2nd-order term involving the gravitational wave field $h_{\mu\nu}$. It suggests that we should introduce a definition:

$$T^{\rm GW}_{\alpha\beta} = -\frac{1}{8\pi G}\left\langle G^{(2)}_{\alpha\beta}\left[h_{\mu\nu};\hat{g}_{\mu\nu}\right]\right\rangle \ . \tag{17.44}$$

A straightforward but lengthy calculation (see Isaacson for details) shows that

$$T^{\rm GW}_{\alpha\beta} = \frac{1}{32\pi G}\left\langle\hat{\nabla}_\alpha\bar{h}_{\mu\nu}\hat{\nabla}_\beta\bar{h}^{\mu\nu} - \frac{1}{2}\hat{\nabla}_\alpha\bar{h}\hat{\nabla}_\beta\bar{h} - \hat{\nabla}_\alpha\bar{h}_{\beta\gamma}\hat{\nabla}_\mu\bar{h}^{\mu\gamma} - \hat{\nabla}_\beta\bar{h}_{\alpha\gamma}\hat{\nabla}_\mu\bar{h}^{\mu\gamma}\right\rangle \ . \tag{17.45}$$

Putting this into TT gauge simplifies things tremendously:

$$\boxed{T^{\rm GW}_{\alpha\beta} = \frac{1}{32\pi G}\left\langle\hat{\nabla}_\alpha h^{\rm TT}_{\mu\nu}\hat{\nabla}_\beta h^{\mu\nu}_{\rm TT}\right\rangle} \tag{17.46}$$

This result is known as the Isaacson stress-energy tensor.

17.4 The power carried by quadrupolar gravitational waves

Let's examine what this tensor tells us about the energy flux carried by quadrupolar gravitational waves in a nearly flat region. By working in this region and choosing nearly flat coordinates, $\hat{\nabla}_\alpha \to \partial_\alpha$, and $h^{\rm TT}_{\mu\nu} \to h^{\rm TT}_{ij}$. We look at T_{00}, which tells us about the flux of energy in the timelike direction:

$$T_{00} = \frac{dE}{dA\,dt} = \frac{1}{32\pi G}\left\langle\partial_t h^{\rm TT}_{ij}\partial_t h^{\rm TT}_{ij}\right\rangle \ . \tag{17.47}$$

The total rate of energy carried by these waves can be found by integrating over a large sphere:

$$\frac{dE}{dt} = r^2\int d\Omega\, T_{00} \ . \tag{17.48}$$

We insert

$$h^{\rm TT}_{ij} = \frac{2G}{r}\frac{d^2 I_{kl}}{dt^2}\left(P_{ki}P_{lj} - \frac{1}{2}P_{kl}P_{ij}\right) \ , \tag{17.49}$$

where the projection tensors are given by $P_{ij} = \delta_{ij} - n_i n_j$. Performing these integrals is facilitated by using the identities

$$\int d\Omega\, n_i n_j = \frac{4\pi}{3}\delta_{ij} \ , \quad \int d\Omega\, n_i n_j n_k = 0 \ , \quad \int d\Omega\, n_i n_j n_k n_l = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta jk\right) \ . \tag{17.50}$$

(You can demonstrate the validity of these identities by choosing a particular coordinate system in which, $n_x = \sin\theta\cos\phi$, $n_y = \sin\theta\sin\phi$, $n_z = \cos\theta$. It is then straightforward to integrate these forms up.) Putting all these pieces together yields

$$\boxed{\frac{dE}{dt} = \frac{G}{5}\left\langle\frac{d^3 I_{ij}}{dt^3}\frac{d^3 I_{ij}}{dt^3}\right\rangle} \tag{17.51}$$

This result is also known as the quadrupole formula, though it is more properly the quadrupole power formula (in contrast to our earlier quadrupole amplitude formula). Using it, one can derive the rate at which a binary system's members fall toward one another due to the backreaction of gravitational wave emission. The result you get works describes quite well the characteristics of many of the sources that are measured by detectors like LIGO today. (There are corrections to this because multipoles higher than the quadrupole also radiate; and, it can be tricky to properly define and compute the multipole moments enter these radiation formulas. On problem set 8, you will compute what you get if you compute the quadrupole moment using a weak-field description of the source.)

17.5 More on the energy and momentum carried by gravitational waves

This section is based quite heavily on Chapters 6 and 12 of *Gravitation* by Eric Poisson and Clifford M. Will, especially sections 6.1, which introduces and discusses the Landau-Lifshitz pseudotensor, and section 12.2, which discusses how to compute radiative backreaction. Interested students are directed to these sections of that book for further discussion.

This Isaacson stress-energy tensor derived above is a nice conceptual tool, particularly for demonstrating that the notion of (for example) energy carried by gravitational waves can be made consistent with the principle of equivalence only by introducing a separation of lengthscales and timescales. One must be able to unambiguously separate the oscillatory, short-scale behavior associated with a wave from the slowly varying, long-scale behavior associated with a background. If such a separation can be made, then the Isaacson tensor provides an object that is rigorously a tensor, at least up to errors that scale as the ratio of lengthscales. This tensor demonstrates that notions of quantities like "energy carried by a wave" must be quadratic in a wave's amplitude, and must incorporate a notion of averaging over some finite region in order to wash out the small-scale oscillations. What's particularly nice is that, because it is a tensor (up least up to errors that scale as the ratio of lengthscales or ratio or timescales), it makes it unambiguously clear that a rigorous notion of energy and momentum carried by gravitational waves exists.

Although I like the Isaacson tensor as a conceptual tool, it turns out not to be such a useful tool for many practical computations. There are two key issues which make it not so useful in practice:

- First, it turns out to not be adequate for computing the angular momentum carried by radiation. A careful calculation of angular momentum radiated according to the Isaacson tensor would yield zero for most components, because the leading terms which appear to contribute to this calculation in fact cancel out upon integrating over the surface of a large sphere. One needs to include terms to higher order in $1/R$ (where R is distance from the radiating source) in order to recover subleading terms that do not vanish upon integration; this cannot be done with Isaacson's tensor.

- Second, although the Isaacson tensor tells us about the "energy" and "momentum" carried by radiation, it does not define what that "energy" or "momentum" actually **is**. Suppose for example you compute the total flux of energy carried by gravitational waves, $dE^{\rm GW}/dt$. You might imagine that you can use conservation of energy to balance this against the "energy" associated with a source, asserting that $d(E^{\rm source} + E^{\rm GW})/dt = 0$. Enforcing this equation, you would learn how the source energy changes, and hope to infer from this change how the system evolves in response.

  The problem is that there is no unique way to define $E^{\rm source}$: one can define "the" energy of a gravitational system in multiple ways in general relativity. Any and indeed, typically *all*, of these definitions may be acceptable and useful, depending on your analysis. The Isaacson tensor does not tell us which notion of "energy" is actually being carried off by gravitational waves. There are some limits for which this this energy is unambiguous (such as the weak-field example you will study in problem set 8), but it is ambiguous in general.

So, how do we do an analysis of this type in a more general situation? The trick is define *a* notion of energy (and momentum and angular momentum) which can be applied both to your system as well as the radiation, and to use that notion consistently throughout the calculation. Such a calculation is *not* covariant, but as long as everything is done consistently, that is fine. One can obtain covariant results that unambiguously describe measurable quantities provided one uses the non-covariant ingredients in the analysis in a fully self consistent way.

One very useful way of defining such quantities is to use what is called the *Landau-Lifshitz pseudotensor*, a quantity defined in the classic text *The Classical Theory of Fields* by Landau and Lifshitz. Interested students are directed to Chapters 6 and 12 of the text by Poisson and Will for a very readable discussion to get further details; those with particular stamina and algebraic fortitude are directed to Landau and Lifshitz itself for detailed proofs of many of the statements I will sketch below.

Landau and Lifshitz reformulate general relativity in an interesting way such that the field equation becomes

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = 16\pi G(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}) \; . \tag{17.52}$$

Here g is the determinant of the metric $g_{\alpha\beta}$. Notice that only *partial* derivatives appear in this equation. The four-index field on the left-hand side is given by

$$H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu} \; , \tag{17.53}$$

where the so-called "gothic metric" is $\mathfrak{g}^{\alpha\beta} \equiv \sqrt{-g}g^{\alpha\beta}$. The field $H^{\alpha\mu\beta\nu}$ is designed so that it has the same symmetries under index exchange as the Riemann tensor.

This reformulation of the field equation is *exact*: no approximation has been introduced. Its main "deficit," so to speak, is that it is not a covariant formulation: the quantities on both sides are tensor densities (note the factors of g) rather than tensors. This is not a flaw, but it's a characteristic that we must bear in mind as we interpret its results. It means (for example) that if we change representations, we must be careful how we interpret the components of terms in this equation, or of quantities that we derive from these terms.

In this formulation of general relativity, the two-index field $t^{\alpha\beta}_{\rm LL}$ is known as the Landau-Lifshitz pseudotensor. It is built from a fairly lengthy sequence of products of partial derivatives of the gothic metric, contracted by various combinations of the "normal" metric. See Eq. (6.5) of Poisson and Will for details. One reason why this is nice is that one can show rather simply that

$$\partial_\beta \left[(-g)\left(T^{\alpha\beta} + t^{\alpha\beta}_{\rm LL}\right)\right] = 0 \; . \tag{17.54}$$

Proof: Take the ∂_β derivative of Eq. (17.52). The derivative operators are symmetric under exchange of β and ν; the field $H^{\alpha\mu\beta\nu}$ is antisymmetric under this exchange. By contracting an antisymmetric field with a symmetric operator, the left-hand side is zero; therefore, the right-hand side must be zero as well.

This derivative of the stress-energy tensor and the Landau-Lifshitz pseudotensor equation expresses local conservation of energy. Here, the stress energy tensor $T^{\alpha\beta}$ as usual describes the flow of 4-momentum through spacetime. The Landau-Lifshitz pseudotensor can then be interpreted — though be *very* cautious about this interpretation — as something like energy and momentum associated with gravity. The reason to be cautious is that this formulation is quite specific to the reformulation of the field equations developed by Landau and Lifshitz. Also, because these are pseudotensors, the notion of "gravitational energy" at some location can be made to vary arbitrarily by changing coordinates. Indeed, one can *always* make $t^{\alpha\beta}_{\rm LL} = 0$ simply by going to the nearly inertial representation of a freely-falling frame. If anyone corners you at a cocktail party[7] and

[7] Or, perhaps more realistically, in an elevator at an APS meeting.

begins expounding on their method for uniquely defining gravitational energy, smile politely and move away as quickly as possible[8].

With all these caveats in mind, the Landau-Lifshitz pseudotensor turns out to be an excellent tool for many applications in which one is interested in understanding how gravitational waves backreact on a source. For example, the flux of energy carried by gravitational waves can be written

$$\frac{dE}{dt} = \oint_{\partial V} (-g) t^{0k}_{\rm LL} dS_k \;, \tag{17.55}$$

where the integral is taken over a large sphere bounding the volume V that surrounds a source, and where dS_k is an (outward directed) area element on that sphere. When evaluated in nearly flat spacetime far from a source, this yields results that agree with the Isaacson formula. Importantly, the Landau-Lifshitz formulation allows us to define what E actually means: the total E contained in a volume V is given by

$$E = \int_V (-g)\left(T^{00} + t^{00}_{\rm LL}\right) d^3x' = \frac{1}{16\pi} \oint_{\partial V} \partial_j H^{0j0k} dS_k \;. \tag{17.56}$$

The second form invokes Eq. (17.52) and uses integration by parts, assuming that V is large enough that a surface term's contribution is negligible. (In practice, one takes V to be all of space, so the bounding surface ∂V is a sphere of infinite radius.) In this formulation, we can compute E from the above integral, and use the equation for dE/dt to infer how E changes due to the backreaction of gravitational-wave emission. Similar formulas allow us to define the momentum and angular momentum of a system, and to infer how these quantities change due to backreaction.

The abundance of tools to describe how gravitational radiation carries energy and backreacts on a system is symptomatic of the fact that general relativity's coordinate freedom means we cannot uniquely define notions of "the" energy associated with gravitational systems. This should be regarded as a feature of the theory; understanding the conceptual nuances associated with defining a system's "energy" and the rate at which this "energy" changes typically yields dividends.

For the purposes of 8.962, the major takeaway is that one should be aware that there exist different approaches to problems like backreaction, and one has freedom to develop and pick tools which are well adapted to your problem. Landau and Lifshitz's reformulation of general relativity is quite elegant, and defines really useful tools, but experience has shown that it works best when the gothic metric $\mathfrak{g}^{\alpha\beta}$ can be regarded as a fairly small deviation from $\eta^{\alpha\beta}$. In this context, it makes it possible to define an iterative scheme for analyzing such a deviation. This approach is an important foundation for post-Newtonian and post-Minkowskian studies of gravitating systems. When applied to systems for which $\mathfrak{g}^{\alpha\beta}$ is not a small deviation from $\eta^{\alpha\beta}$, this reformulation is quite cumbersome and difficult to use. Studying the strong-field spacetimes of black holes in this framework, for example, is rather nightmarish.

[8]Whatever you do, do *not* attempt to correct their misunderstandings. Noble as your impulse may be, nothing good can come of it.