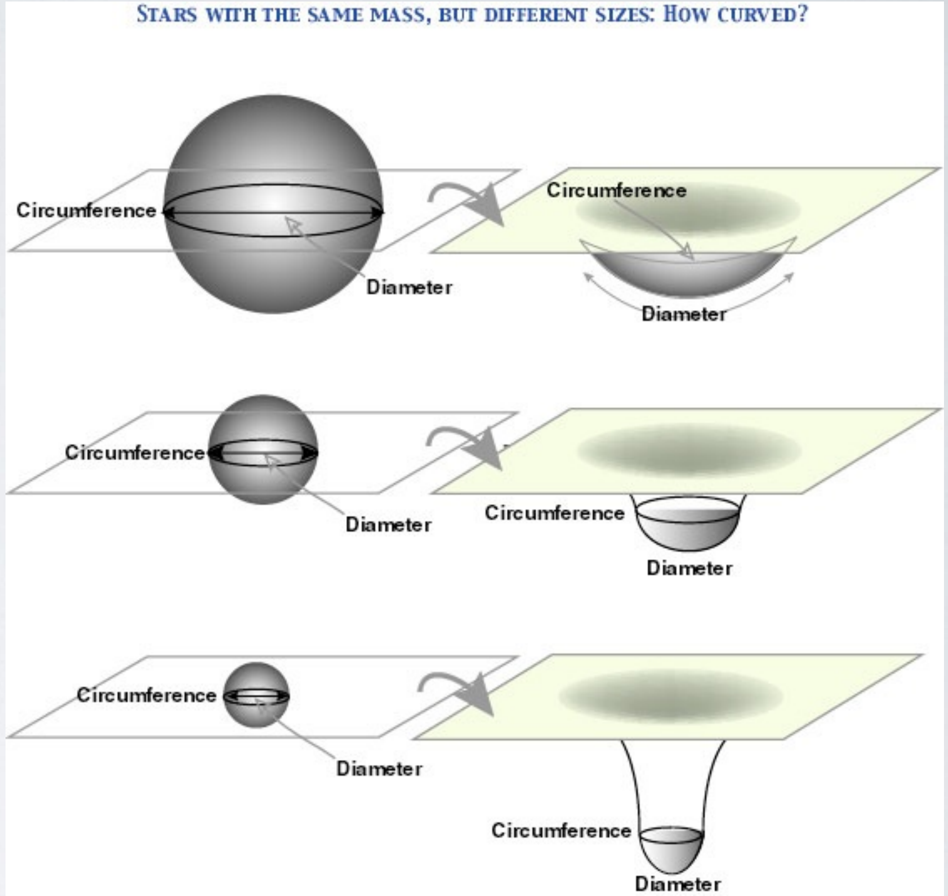
8.962 LECTURE 20

Spherical symmetry I



Cartoon of spacetime curvature for a sequence of progressively more compact stars.

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Recap: After determining that the "best" line element for the largest scales in the universe takes the form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right]$$

with the scale factor determined by the Einstein field equation, yielding the Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 \equiv H(a)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2} \qquad\qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3P\right)$$

$$\partial_t\left(\rho a^3\right) = -P\,\partial_t\left(a^3\right)$$

Define the density parameter

$$\rho_{\rm crit} = \frac{3H^2}{8\pi G}\,, \qquad \Omega = \rho/\rho_{\rm crit}$$

Challenge is then to find a way to observe the *a* at many values of *t*, learn from it how the universe behaves.

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Two pieces to this: First, scale factor is encoded in the *redshift z* of radiation we measure. If something is emitted at scale factor *a* and is measured now (when the scale factor is 1), then

$$a = \frac{1}{1+z}$$

This lets us associate scale factor with objects we study. Also need time *t* that goes with the scale factor; since light travels on null paths, we use distances. Angular diameter distance given by

$$D_a = \frac{1}{(1+z)H_0}\mathcal{S}(\Omega_{c0}, \Omega_x, z, k)$$

Other notions of distance simply related by redshift factor

$$D_L = (1+z)D_M = (1+z)^2 D_A$$

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The factor $\mathcal{S}(\Omega_{c0}, \Omega_x, z, k)$ is an integral given by

$$\begin{aligned}
\mathcal{S} &= |\Omega_{c0}|^{-1/2} \sin\left[|\Omega_{c0}|^{1/2}\int_0^z \frac{dz'}{E(z')}\right] \qquad k = 1 \\
&= \int_0^z \frac{dz'}{E(z')} \qquad k = 0 \\
&= |\Omega_{c0}|^{-1/2} \sinh\left[|\Omega_{c0}|^{1/2}\int_0^z \frac{dz'}{E(z')}\right] \qquad k = -1
\end{aligned}$$

where

$$E(z) = \left[\sum_i \Omega_{i0}(1+z)^{n_i}\right]^{1/2}$$

Measure distances and redshifts for many objects; fit cosmological models, determine parameters H_0, various Ωs.

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Publication history:

- 18 November 1915: Einstein's overview of the theory of general relativity published by the Prussian Academy of Sciences
- 25 November 1915: Einstein publishes how to use the theory to compute the perihelion precession of Mercury
- 2 December 1915: Einstein publishes in detail the field equations of general relativity
- 22 December 1915: Letter from Karl Schwarzschild to Albert Einstein announces the exact solution we now call the Schwarzschild solution: "**As you see, the war treated me kindly enough, in spite of the heavy gunfire, to allow me to get away from it all and take this walk in the land of your ideas.**"

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Result from old homework exercise: You showed that

$$\nabla_\mu T^{\mu\nu} = 0$$

when applied to a fluid in hydrostatic equilibrium leads to the condition

$$(\rho + P)u^\beta \nabla_\beta u_\alpha = -\partial_\alpha P - u_\alpha u^\beta \partial_\beta P$$

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For our spacetime, $u^\alpha \doteq (e^{-\Phi(r)}, 0, 0, 0)$, and this becomes

$$(\rho + P)\Gamma^t{}_{rt} = -\frac{dP}{dr}\ ; \quad \Gamma^t{}_{rt} = \frac{d\Phi}{dr}$$

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Combine with Einstein equation result for $d\Phi/dr$:

$$\frac{dP}{dr} = -\frac{G(\rho + P)(m(r) + 4\pi r^3 P(r))}{r(r - 2Gm(r))}\ .$$