Last time: Determined that the spacetime of a spherically symmetric body is given by

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - 2Gm(r)/r} + r^2 d\Omega^2$$

using "Schwarzschild coordinates", and where the body has

$$\begin{array}{ll} P = P(r) & \\ \rho = \rho(r) & r \le R_* \\ \rho = P = 0 & r > R_* \end{array}$$

Enforcing vacuum Einstein (appropriate for $r > R_*$) leads to

$$m(r) = m(R_*) \equiv M_{TOT}$$

$$e^{2\Phi(r)} = 1 - 2GM_{TOT}/r$$

Enforcing Einstein in the interior leads to

$$m(r) = 4\pi \int_0^r \rho(r')(r')^2 dr'$$

$$\frac{d\Phi}{dr} = \frac{G(m(r) + 4\pi r^3 P(r))}{r(r - 2Gm(r))}$$

Enforcing $\nabla_\mu T^\mu{}_r = 0$ leads to

$$\frac{dP}{dr} = -\frac{G(\rho + P)(m + 4\pi r^3 P)}{r(r - 2Gm)}$$

"TOV equations". Non-rel, Newtonian limit:

$$\frac{d\Phi}{dr} = \frac{Gm(r)}{r^2}$$

$$\frac{dP}{dr} = -\frac{G\rho(r)m(r)}{r^2}$$

Highly idealized, unrealistic - but instructive! - limit: ρ = constant.

"Star" with sound speed, $c_s^2 = dP/d\rho = \infty$! Not physical.

Trivial mass function:

$$m(r) = \frac{4}{3}\pi\rho r^3 \qquad r \le R_*$$

$$= \frac{4}{3}\pi\rho R_*^3 \qquad r > R_*$$

Pressure profile much more complicated:

$$\frac{dP}{dr} = -\frac{4}{3}\pi r G\left[\frac{(\rho+P)(\rho+3P)}{1-8\pi G\rho r^2/3}\right]$$

Miracle: This has a simple solution.

$$\left[\frac{\rho+3P}{\rho+P}\right] = \left[\frac{\rho+3P_c}{\rho+P_c}\right]\left(1-\frac{2Gm(r)}{r}\right)^{1/2}$$

where $P_c = P[r=0]$ - pressure at center.

Using the fact that $P \to 0$ at $r = R_*$, can relate stellar radius to choice of central pressure:

$$R_*^2 = \frac{3}{8\pi G\rho}\left[1-\frac{(\rho+P_c)^2}{(\rho+3P_c)^2}\right]$$

-or-

$$P_c = \frac{\rho\left[1-(1-2GM_{TOT}/R_*)^{1/2}\right]}{3\sqrt{1-\frac{2GM_{TOT}}{R_*}}-1}$$

Have a one parameter family of models:

Pick P_c, get R_* ... or vice versa.

Formula for P_c diverges for a certain "compactness":

$$P_c \to \infty \quad \text{as} \quad 3\sqrt{1 - \frac{2GM_{TOT}}{R_*}} - 1 = 0$$

$$1 - \frac{2GM_{TOT}}{R_*} = \frac{1}{9}$$

$$\longrightarrow \quad \boxed{\frac{GM_{TOT}}{R_*} = \frac{4}{9}}$$

Solutions imply a "maximum compactness": For uniform density stars, we <u>cannot</u> have physically reasonable pressure profiles if $\frac{R_*}{GM} < \frac{9}{4}$.

This limit holds in general - known as Buchdahl's theorem: <u>No</u> stable, spherical fluid configuration can exist if $R_* < 9GM_*/4$. Only requirement: Need $dP/dr < 0$ over star. See Weinberg, Sec 11.6.

What if such a star existed? It would not be stable! Need to consider time evolution ... to be discussed soon.

Real stars: Not constant density!

general case: $P = P(\rho)$

Even more general case: $P = P(\rho, S)$ where S = entropy.

For applications in which general relativity is important - e.g., structure of neutron stars - the fluid is often so cold that entropy decouples: via $dU = -PdV + TdS$, can ignore entropy.

What does cold mean? Depends on situation! For neutron stars, we are dealing with a Fermi fluid, so relevant scale is set by Fermi temperature:

$$T_F = \frac{E_F}{k_B} = \frac{\sqrt{p_F^2 + m^2c^4}}{k_B}$$

$$p_F = \left[\frac{\rho h^3}{8\pi m}\right]^{1/3}$$

Plug in numbers appropriate for a neutron star: $\rho \sim 10^{16}$ gm/cm^3, $m = m_N \rightarrow T_F \sim 10^{13}$ K.

Observations: $T \sim 10^6 - 10^9$ K $\ll T_F$

$\hookrightarrow$ supernova!

So - cold! At least for this example.

An approximation that holds in certain limits, useful for our purposes: Pressure is a power law as a function of density:

$$P = K\rho^{\Gamma}$$

K, Γ are constants. Form known as "polytrope." At low densities, $\Gamma = 5/3$ has important role in astrophysics (non-~~[illegible]~~ relativistic electron degeneracy pressure; fully convective ideal gas law.) $\Gamma = 2$ gives a good fit to many neutron-star models.

Recipe to build a stellar model:

1. Pick $\rho(r=0) \equiv \rho_c$; implies P_c
2. Set $m(r) = 0$ @ $r = 0$.
3. Integrate dP/dr, $m(r)$ outward from $r=0$ using a numerical integrator. Optional: also integrate $d\Phi/dr$ from center. Its initial value is not known - set it to zero for now.
4. When you hit $P=0$, that's the surface of the star: $P(r) = 0$ defines $r = R_*$. You now know the star's mass and radius.

Boundary condition: By Birkhoff's theorem, we know that $\Phi = \frac{1}{2}\ln(1 - 2GM/r)$ for $r \geq R_*$.

So, adjust interior solution for Φ by a constant so that $\Phi(R_*) = \frac{1}{2}\ln(1 - 2GM/R_*)$.

Consider spacetime that is Schwarzschild

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

for <u>all</u> r, not just the exterior of some object.

This is an exact, <u>VACUUM</u> solution - but it has orbits / geodesics indicating it has a mass M!

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta = 0 \xrightarrow[\text{field}]{\text{weak}} \frac{d^2 x^i}{d\tau^2} + \Gamma^i{}_{00} u^0 u^0 = 0$$

$$\rightarrow \quad \frac{d^2 x^i}{dt^2} = \frac{\partial}{\partial x^i}\left(\frac{GM}{r}\right)$$

What is vacuum ... but has a mass M?

Analogous to Coulomb point charge:

$$\underset{\sim}{\bar{E}} = \frac{q \bar{x}}{r^3} \rightarrow \underset{\sim}{\nabla} \cdot \underset{\sim}{E} = 4\pi\rho = 0$$

No density ... but total charge must be q.

Resolution: Singular point charge at $r=0$.

Expect similar resolution for our case: Schwarzschild solves $G_{\mu\nu} = 0$ everywhere ... but $r=0$ might be a bit odd.

(Even more odd than in Maxwell case thanks to nonlinearities!)

Now look at spacetime itself. Two radii look troublesome:

$r = 2GM$, $r = 0$.

Metric components can be deceiving, so examine curvature (Carroll, Eq 5.13 gives non-zero Riemann components).

From this, an invariant curvature measure:

$$I \equiv R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} \equiv \text{"Kretschmann scalar"}$$

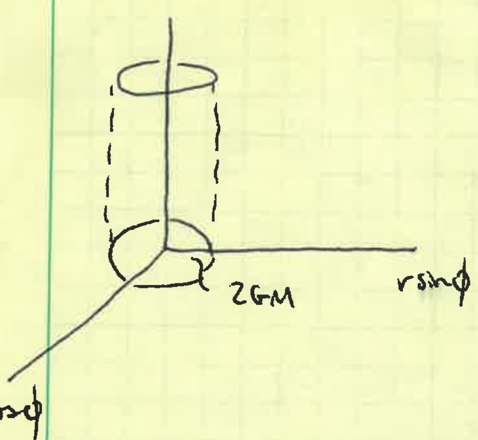
$$= \frac{48 G^2 M^2}{r^6}$$

$\sqrt{I}$ is roughly an invariant characterization of tidal forces felt by a body in the spacetime.

Notice: <u>NOT</u> singular at $r = 2GM$, but singular at $r = 0$.

→ Reminiscent of Coulomb point charge!

So, what is $r = 2GM$? Consider the circle $r = 2GM$, $\theta = \pi/2$ as time advances: what is the spacetime area of the tube this circle sweeps out?



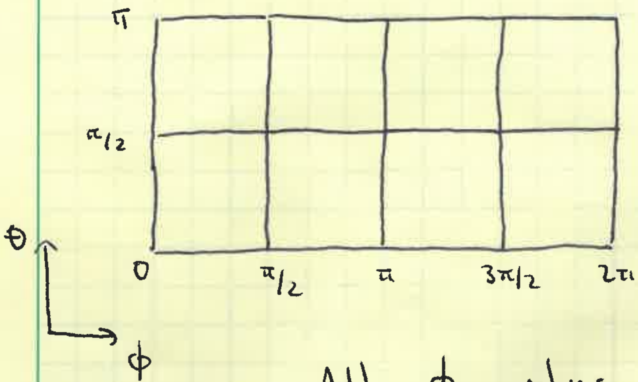
$$A_{tube} = \int_{t_{start}}^{t_{end}} dt \int_0^{2\pi} d\phi \times \left[g_{tt}\, g_{\phi\phi} \right]^{1/2}_{r=2GM,\ \theta=\pi/2}$$

$$= 2GM \int dt \int_0^{2\pi} d\phi \left[1 - \frac{2GM}{r} \right]^{1/2}_{r=2GM}$$

$$= 0$$

"World tube" has no surface area!

Issue is that coordinate t is pathological there! Consider analogy: Draw a sphere as follows:



Perfectly accurate rendering of a sphere's coordinates ... but a <u>horrible</u> rendering of its <u>geometry</u>.

All ϕ values at $\theta = 0$, $\theta = \pi$ occupy a single point. This drawing tempts us to compute quantities like "length" of the $\theta = 0$ slice of the sphere.

Tube drawing likewise represents Schwarzschild coordinates, but distorts the geometry. In fact, <u>all</u> times t map to a single sphere at $r = 2GM$!

Insight into what happens near that radius requires us to think carefully about the nature of the time parameter.

Example: Drop particle from rest at $r = r_s$; integrate geodesic equation to find its motion as a function of "time".

"time" = proper time as measured by the body, τ

= coordinate time t

Result:
$$\tau = \frac{r_s^{3/2}}{\sqrt{2GM}} \tan^{-1}\left(\sqrt{\frac{r_s - r}{r}}\right) + \sqrt{\frac{r r_s (r_s - r)}{2GM}}$$

$$t = 2GM \ln\left[\frac{\sqrt{r\left(\frac{r_s}{2GM} - 1\right)} + 1}{\sqrt{r\left(\frac{r_s}{2GM} - 1\right)} - 1}\right]$$
$$+ \sqrt{r(r_s - r)\left(\frac{r_s}{2GM} - 1\right)}$$
$$+ (r_s + 4GM)\sqrt{\frac{r_s}{2GM} - 1}\left[\frac{\pi}{2} - \tan^{-1}\left(\sqrt{\frac{r}{r_s - r}}\right)\right]$$

From this, see $\tau \to \frac{\pi r_0^{3/2}}{2\sqrt{2GM}}$ as $r \to 0$

But as $r \to 2GM$, $t \to \infty$!

From line element, t is time used by distant observers. So infalling observer reaches $r = 0$ in finite time on their own clocks -- but never reaches $r = 2GM$ according to distant observers.