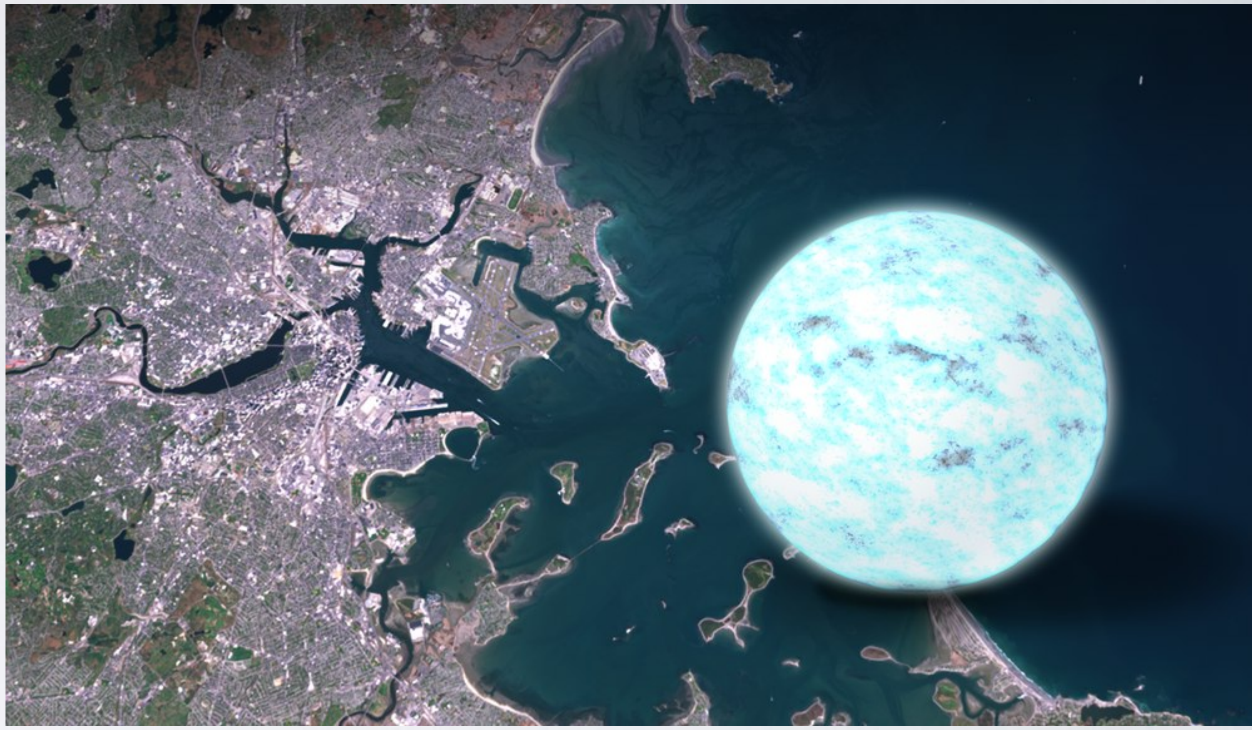
8.962 LECTURE 21

Spherical symmetry II:
How to build ultra-dense objects;
limitations on their existence



Neutron star versus Boston Harbor (artist's impression, from wikimedia commons, produced by Goddard Space Flight Center)

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Recap: Metric of a spherically symmetric body:

$$ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - 2Gm(r)/r} + r^2 d\Omega^2$$

In "Schwarzschild coordinates." The body has density and pressure $\rho = \rho(r), P = P(r)$ for all radii $r \le R_*$; these functions go to zero for $r > R_*$. Enforcing the vacuum Einstein equations in the exterior (vacuum) of this object leads to

$$m(r) = M \equiv M_{\text{tot}}$$

$$e^{2\Phi(r)} = 1 - \frac{2GM_{\text{tot}}}{r}$$

The mass M_{tot} introduced is often called the "gravitational" mass of the body: it is the mass we infer from measurements of the body's gravity (e.g., with radial infall, or orbits [to be discussed soon]).

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If we enforce Einstein in the interior of the body we find

$$m(r) = 4\pi \int_0^r \rho(r')(r')^2 dr'$$

$$\frac{d\Phi}{dr} = \frac{G[m(r) + 4\pi r^3 P(r)]}{r[r - 2Gm(r)]}$$

Note that this *m*(*r*) is a **definition**: the integral is not over a *proper* volume element of the body's interior! Further enforcing $\nabla_\mu T^{\mu\nu} = 0$ in the interior leads to an equation for the pressure:

$$\frac{dP}{dr} = -\frac{G[\rho(r) + P(r)][m(r) + 4\pi r^3 P(r)]}{r[r - 2Gm(r)]}$$

System [*dP*/*dr*, *d*Φ/*dr*, *m*(*r*)] is known as the *Tolman-Oppenheimer-Volkov* equations.

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Radial infall: Using equations we will discuss next week, can obtain $dr/d\tau$ (radial component of geodesic 4-velocity) and dr/dt (ratio of radial and time components of geodesic 4-velocity) for a body that falls from rest, starting at $r = r_s$ at $t = \tau = 0$:

$$\frac{dr}{d\tau} = -\sqrt{\frac{2GM}{r} - \frac{2GM}{r_s}}$$

$$\frac{dr}{dt} = -\left(1 - \frac{2GM}{r}\right)\left(1 - \frac{2GM}{r_s}\right)^{-1/2}\sqrt{\frac{2GM}{r} - \frac{2GM}{r_s}}$$

These expressions can be integrated in a way to tell us how much "time" it takes to move from r_s to some general radius r.

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Result of this:

$$\tau = \frac{r_s^{3/2}}{\sqrt{2GM}} \arctan\left(\sqrt{\frac{r_s - r}{r}}\right) + \frac{\sqrt{r r_s (r_s - r)}}{\sqrt{2GM}}$$

$$t = 2GM \ln\left[\frac{\sqrt{\frac{r\left(\frac{r_s}{2GM} - 1\right)}{r_s - r}} + 1}{\sqrt{\frac{r\left(\frac{r_s}{2GM} - 1\right)}{r_s - r}} - 1}\right] + \sqrt{r(r_s - r)\left(\frac{r_s}{2GM} - 1\right)}$$

$$+(r_s + 4GM)\sqrt{\frac{r_s}{2GM} - 1}\left[\frac{\pi}{2} - \arctan\left(\sqrt{\frac{r}{r_s - r}}\right)\right]$$

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Infalling radius versus "time":