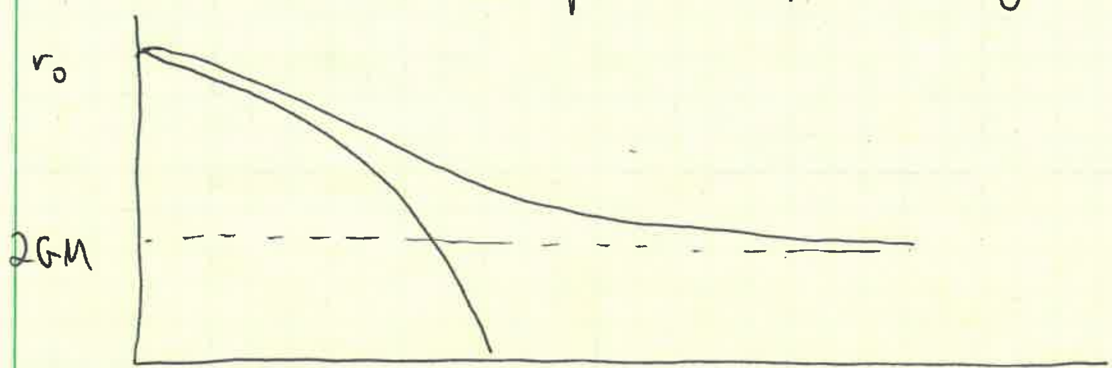
Recap: Considering spacetime that is Schwarzschild everywhere:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

- Vacuum
- Has mass M as defined by free-fall motion
- $r = 0$ tidal singularity
- $r = 2GM$ odd: takes infinite coordinate time to reach it, but finite proper time for infalling observer:



What's going on?

Imagine as the body falls in that it emits a pulse of radiation with some frequency. Far away ($r \ggg 2GM$), we have

$$p_\alpha^{\text{pulse}} = \hbar\omega_\infty\,(-1, 1, 0, 0)$$

where ω_∞ is the frequency measured very far away.

Recall: 1. Energy measured by an observer with $\vec{u}$ is

$$E = -\vec{p}\cdot\vec{u}$$

2. In a time-independent spacetime, p_t = constant.

Static observer in Schwarzschild has

$$u^\alpha \doteq \left[\left(1 - \frac{2GM}{r}\right)^{-1/2}, 0, 0, 0\right]$$

Imagine radiation pulse is generated at $r = R$ with energy $\hbar\omega_R$:

$$\hbar\omega_R = -p_\alpha u^\alpha$$

4-momentum of radiation

4-velocity of static observer at $r = R$

Using all of our pieces, we see that

$$\hbar\omega_R = -p_t\left(1 - \frac{2GM}{R}\right)^{-1/2}$$

Now, let that radiation propagate out to very distant observers ... and take advantage of the fact that, for the light, p_t = constant on its trajectory:

$$\hbar\omega_\infty = -p_t$$

Since $u^\alpha \to (1, 0, 0, 0)$ for static observers as $r \to \infty$.

Putting these together, we see that

$$\boxed{\hbar\omega_\infty = \hbar\omega_R\left(1 - \frac{2GM}{R}\right)^{1/2}}$$

The pulse redshifts away! $R = 2GM$ is a surface of infinite redshift.

Correspondingly, can show that if pulses are spaced by ΔT as measured by observers at $r = R$, then the interval between pulses far away is

$$\Delta T_\infty = \Delta T_R \left(1 - \frac{2GM}{R}\right)^{-1/2}$$

$$\rightarrow \infty \quad \text{as} \quad R \rightarrow 2GM$$

Punchline: As an emitter approaches $r = 2GM$, information from it becomes infinitely redshifted, and takes infinite time to get out.

Recall that when we define time in flat spacetime, we use "Einstein synchronization" procedure to connect separated clocks. The Schwarzschild time coordinate hooks this time into a region of strong gravity.

Evidently, something goes awry as we approach $r = 2GM$! The light rays which we used to synchronize the spacetime's clocks are themselves behaving pathologically.

Need better coordinates to build a better view of strong gravity spacetime. Since pathology is encoded in null geodesics, let's focus on them.

Radial null geodesics:

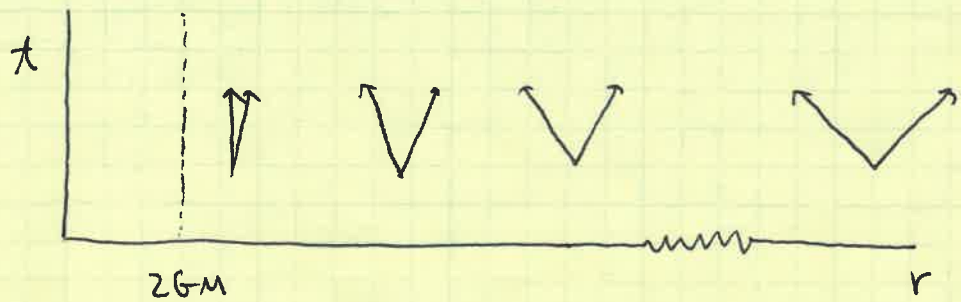
$$0 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1} dr^2$$

$$\rightarrow \quad \frac{dt}{dr} = \pm\left(1-\frac{2GM}{r}\right)^{-1}$$

dt/dr defines the opening angle of "light cones".

Large r: 45° to vertical - intuitive behavior.

$r \to 2GM$: Angle collapses to 0°:



1ST step to better coordinates: Can move pathology from time to radius by choosing "tortoise coordinate":

$$dt = \pm\, dr_* \qquad \text{where}$$

$$r_* = r + 2GM \ln\left[\frac{r}{2GM} - 1\right]$$

In this coord system: light cones are 45° everywhere... but $r = 2GM$ maps to $r_* = -\infty$. Infinitely stretched the strong field region.

We use r_* as an intermediary for designing coords adapted to radiation:

$$v = t + r_* \equiv \text{constant for ingoing radial null rays}$$

$$u = t - r_* \equiv \text{constant for \underline{outgoing} radial null rays.}$$

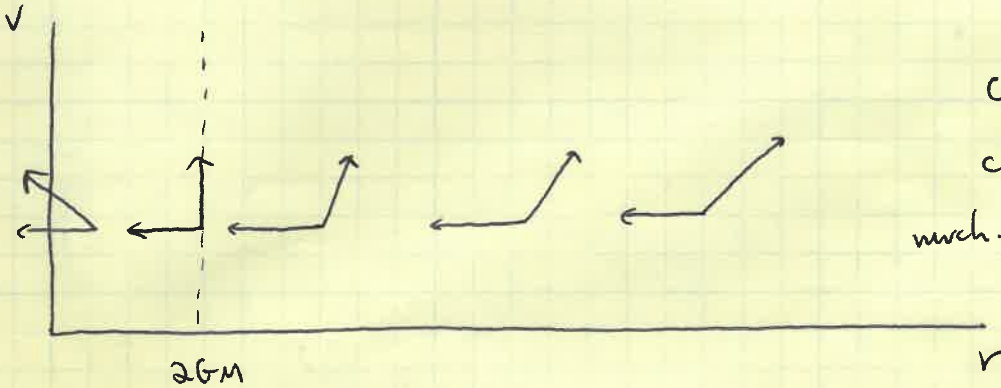
Can choose either v or u to replace Schwarzschild t.
Chose v – yields Schw. metric in "ingoing Eddington-Finkelstein" coordinates:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

Radial null curves are given by

$$\frac{dv}{dr} = 0 \quad \text{(ingoing)}$$

$$= 2\left(1 - \frac{2GM}{r}\right)^{-1} \quad \text{(outgoing)}$$



cones don't close up as much... tip over one side!

(Note: Fig 5.10 of Carroll not quite drawn right.)

All timelike trajectories must lie between sides of the null cone – extreme limits of timelike behavior!

→ At $r \le 2GM$, <u>NO</u> timelike trajectory "gets out": All future directed trajectories must go to smaller radii!

<u>NOTHING</u> that crosses $r = 2GM$ is <u>ever</u> coming back.

This is a <u>HORIZON</u>: no events inside the horizon can have a causal influence on events outside.

Hence, "Event horizon". Spacetimes with mass and event horizons are called <u>Black holes</u>.

Final useful but rather obtuse transformation: Put

$$v' = \exp\left[v/4GM \right]$$

$$u' = -\exp\left[-u/4GM \right]$$

Then define $T = \frac{1}{2}(v'+u')$, $R = \frac{1}{2}(v'-u')$

Simple to show

$$\left.\begin{aligned} T &= \pm\sqrt{\frac{r}{2GM}-1}\; e^{r/4GM} \sinh\left(\frac{t}{4GM}\right) \\ R &= \pm\sqrt{\frac{r}{2GM}-1}\; e^{r/4GM} \cosh\left(\frac{t}{4GM}\right) \end{aligned}\right\} r \geq 2GM$$

$$\left.\begin{aligned} T &= \pm\sqrt{1-\frac{r}{2GM}}\; e^{r/4GM} \cosh\left(\frac{t}{4GM}\right) \\ R &= \pm\sqrt{1-\frac{r}{2GM}}\; e^{r/4GM} \sinh\left(\frac{t}{4GM}\right) \end{aligned}\right\} r \leq 2GM$$

Can invert to original Schwarzschild coordinates via

$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM}$$

$$\begin{aligned} T/R &= \tanh(t/4GM) \qquad r \geq 2GM \\ &= \coth(t/4GM) \qquad r \leq 2GM \end{aligned}$$

(T, R, θ, ϕ) are "Kruskal – Szekeres" coordinates.

Why?? First, get metric that is well-behaved everywhere

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}\left(-dT^2 + dR^2\right) + r^2 d\Omega^2$$

Notice radial null geodesics make 45° trajectories in these coordinates! $ds^2 = 0 \rightarrow dT = \pm dR$.

Second, these coordinates highlight the <u>causal</u> <u>structure</u> of the spacetime: Which events can influence which other events.

Notice: surfaces of constant r form hyperbolae in K-S:

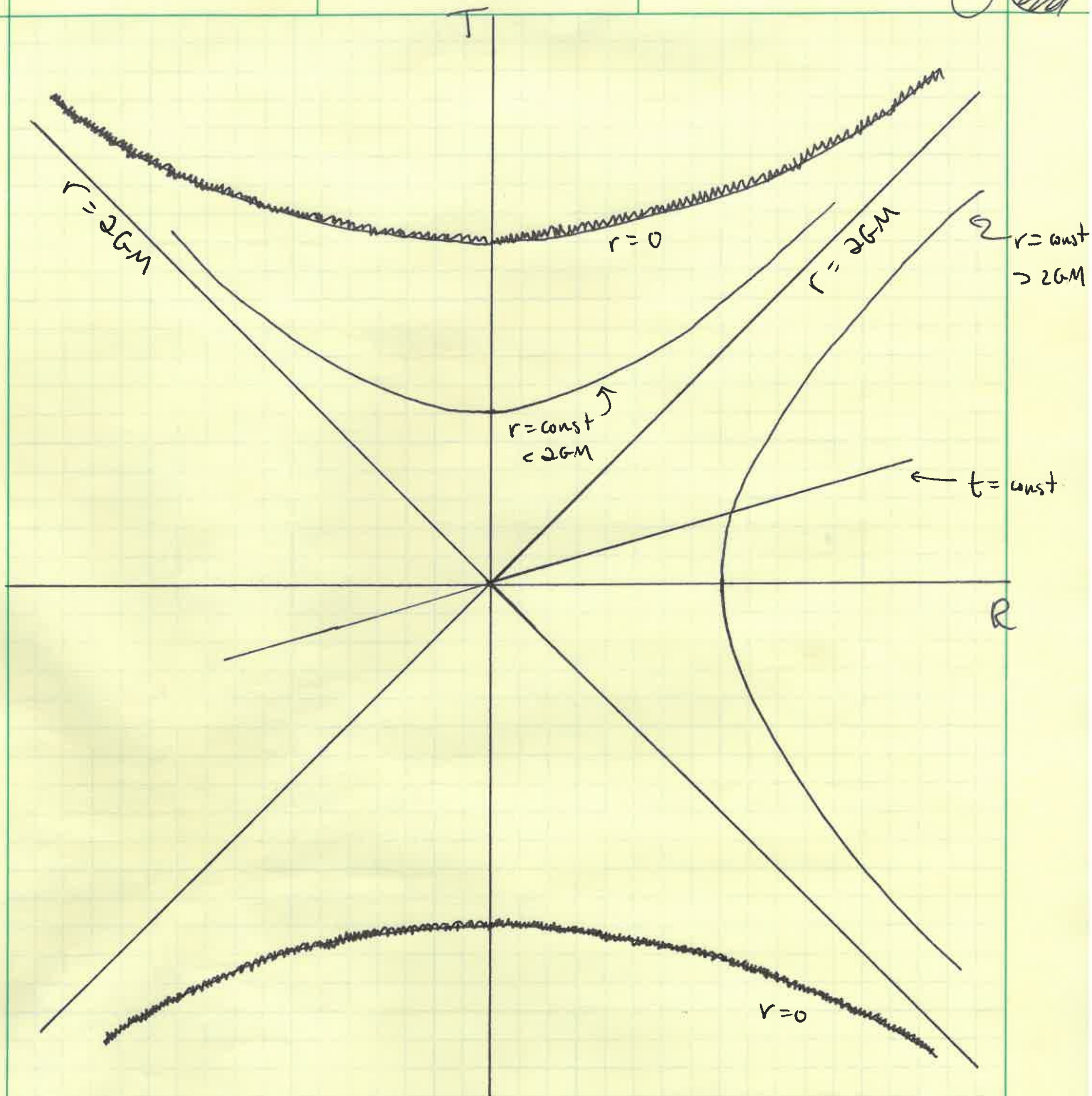
$$T^2 - R^2 = \left(1 - \frac{r}{2GM}\right) e^{r/2GM} = \text{const}$$

Event horizon: $r = 2GM \rightarrow T = \pm R$

Surfaces of constant t form lines:

$$\frac{T}{R} = \tanh\left(\frac{t}{4GM}\right) = \text{const.}$$

Note: $T = \pm R \rightarrow t \rightarrow \pm\infty$!

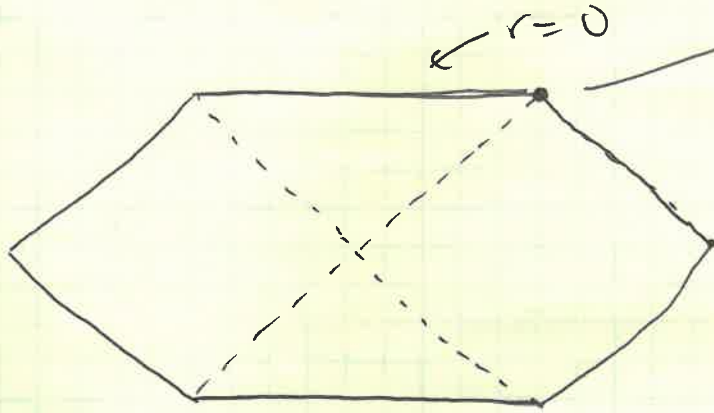


Since speed of light trajectories are 45° lines on this diagram, clear that <u>no</u> physical trajectory can move from $r < 2GM$ to $r > 2GM$. Interior is "causally disconnected."

One final coordinate transformation:

$$R+T = \tan\left[(\psi+\xi)/2\right]$$

$$R-T = \tan\left[(\psi-\xi)/2\right]$$



$(\psi = \pi/2,\ \xi = \frac{\pi}{2})$

$(\psi = 0,\ \xi = \pi)$

↓

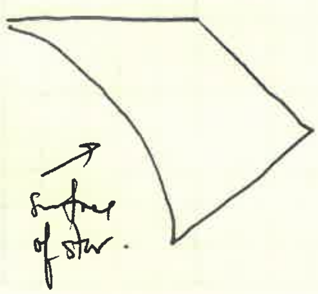
limit of Points for $r \to \infty$, $t \to$ finite.

"Spacelike" Infinity

$\psi = \pi/2,\ \xi = \pi/2$: limit of points for $r \to$ finite, $t \to$ infinity "Time like" Infinity

Between them: NULL infinity.

Penrose diagram. with collapse, we get



Summary:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)dr^2 + r^2 d\Omega^2$$

is a black hole.

Vacuum everywhere ... except for singular field equations at $r=0$.

Bad coordinates at $r = 2GM$: "Surface of infinite redshift"

"Event horizon": Once in, you never get out. All physical trajectories hit the singularity at $r=0$.

Other black holes:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2$$

Comes from $T^{\mu}{}_{\cancel{\mu}\nu} = \mathrm{diag}\left[\frac{-Q^2}{8\pi r^4}, -\frac{Q^2}{8\pi r^4}, \frac{Q^2}{8\pi r^4}, \frac{Q^2}{8\pi r^4}\right]$

→ Black hole metric with a Coulomb electric field!

Horizon is at root of function:

$$r_{horiz} = \cancel{G\left[M + \sqrt{M^2 - Q^2}\right]} \; GM + \sqrt{G^2M^2 - GQ^2}$$

Notice: if ~~|Q| > M~~, no horizon! Still a singularity at $r=0$ - "naked".

↓

$$G^{1/2}|Q| > GM$$