8.962 LECTURE 22

The global structure of the (vacuum) Schwarzschild spacetime

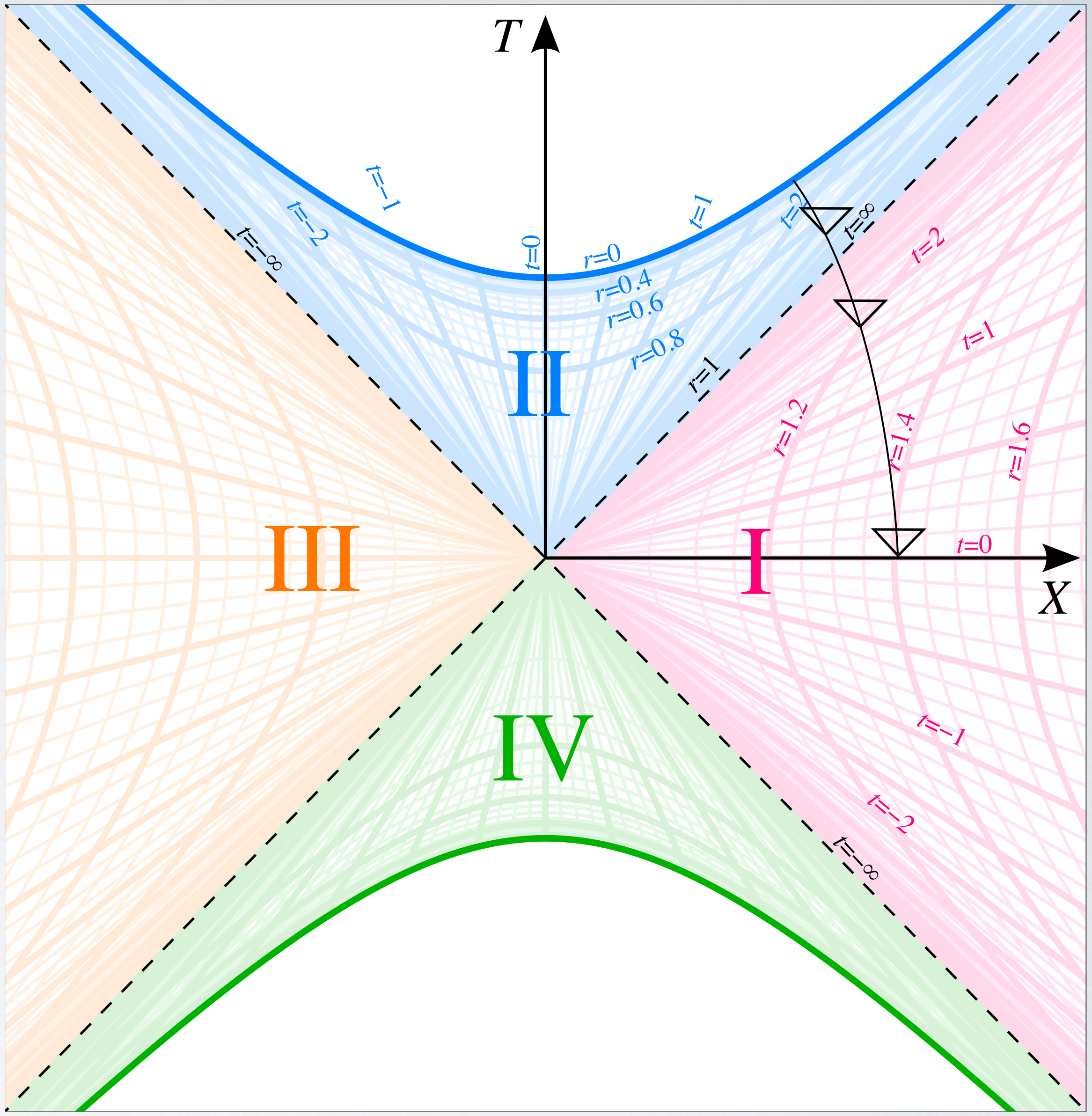

Figure taken from Wikipedia article "Kruskal-Szekeres coordinates," made by Dr Greg, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=22635538

(If you look at it closely, note that Dr Greg has chosen units in which $2GM = 1$.)

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Recap: Globally vacuum Schwarzschild spacetime,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2 d\Omega^2$$

Spacetime for which $T_{\mu\nu} = 0$ everywhere, but it has a gravitational mass of *M*! Examine the "Riemann squared" curvature scalar,

$$I \equiv R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} = \frac{48 G^2 M^2}{r^6}$$

can see that $r = 0$ is a true curvature singularity (more on this later), but there's nothing "noteworthy" about the radius $r = 2GM$.

Examination of surfaces in spacetime shows that a "ring" of radius $r = 2GM$ sweeps out a world tube of spacetime surface area ***zero***; indicates that something weird is going on with time coordinate.

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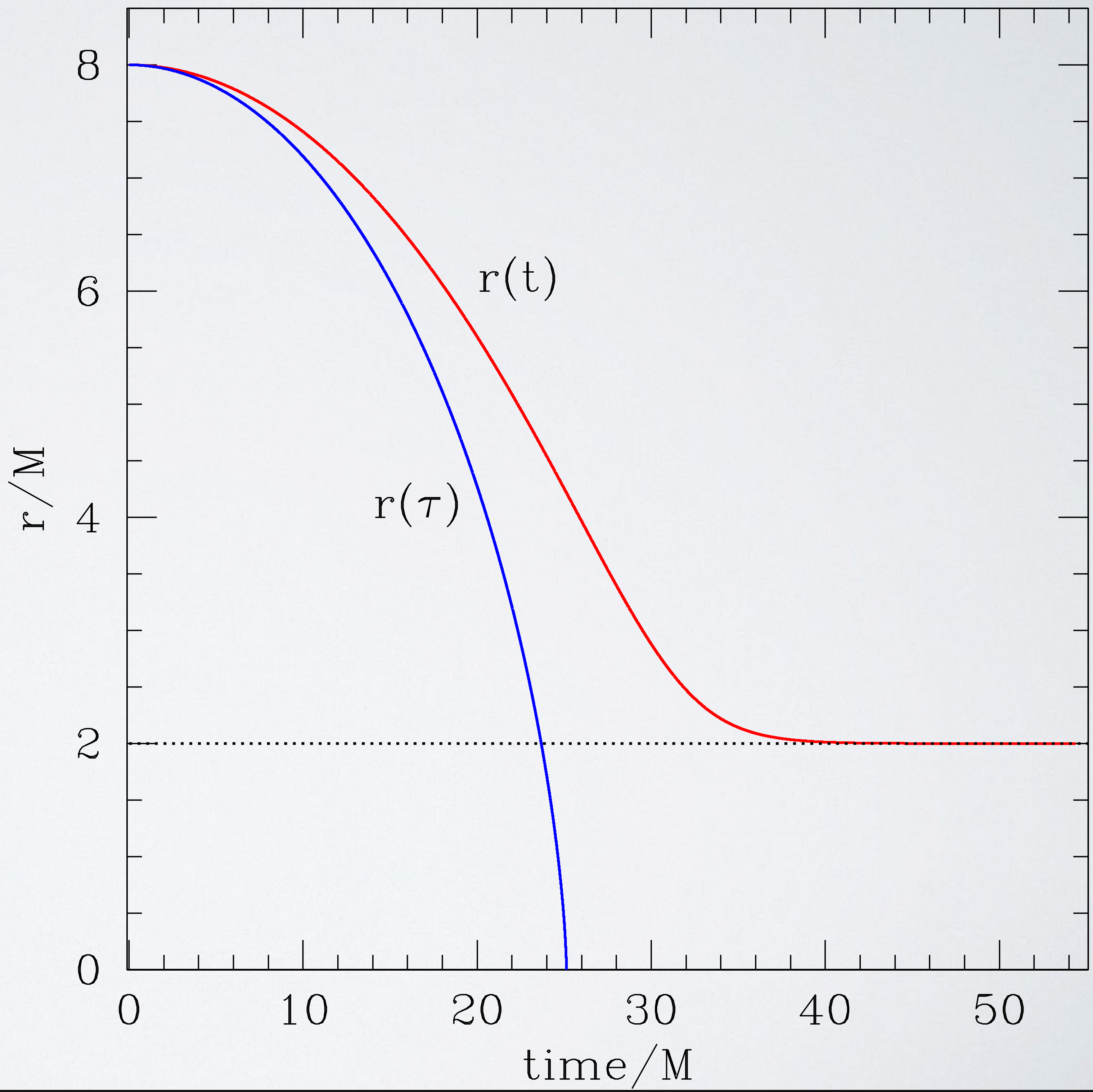
Example: an infalling body's radial coordinate versus "time":

Body reaches $r = 0$ in finite proper time; body asymptotically approaches $r = 2GM$ as a function of coordinate time t.

Coordinate t describes time as measured by very distant observers … so distant observers never "see" infalling body cross the coordinate $r = 2GM$.



8.962 LECTURE 22

Static observer at $r = R$ is not freely falling: motion through spacetime governed by

$$\frac{du^r}{d\tau} + \Gamma^r{}_{\mu\nu} u^\mu u^\nu = a^r$$

To be static, we require $du^r/d\tau = 0$; combine with the 4-velocity and Schwarzschild Christoffel, and find acceleration to stand still is

$$a^r = \frac{GM}{R^2}$$

Looks just like Newtonian! Check its normalization:

$$|\vec{a}| = \sqrt{g_{\mu\nu} a^\mu a^\nu} = \sqrt{g_{rr} a^r a^r}$$

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$$|\vec{a}| = \sqrt{g_{\mu\nu} a^\mu a^\nu} = \sqrt{g_{rr} a^r a^r}$$
$$= \frac{GM}{R^2} \frac{1}{\sqrt{1 - 2GM/R}}$$

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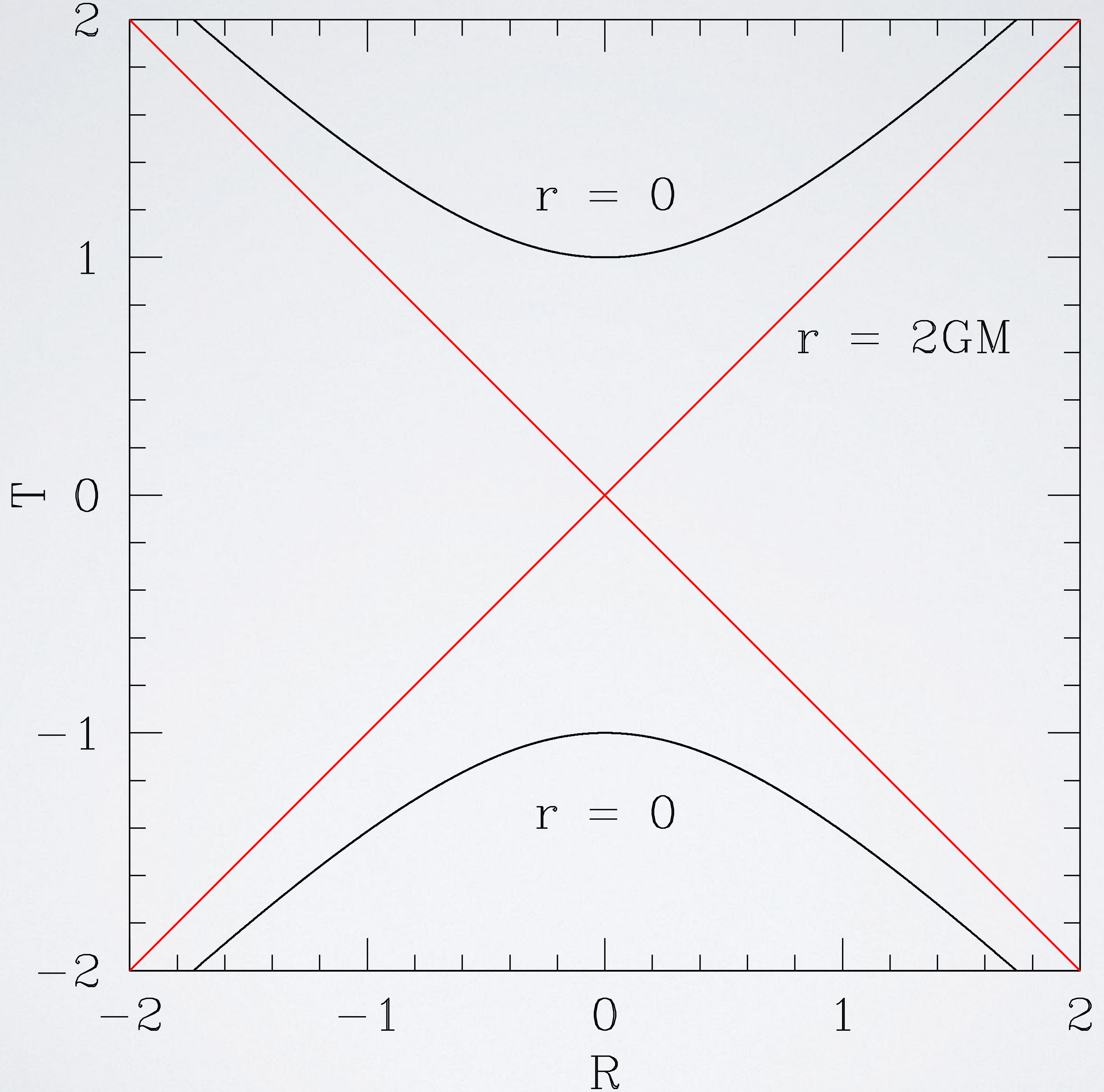

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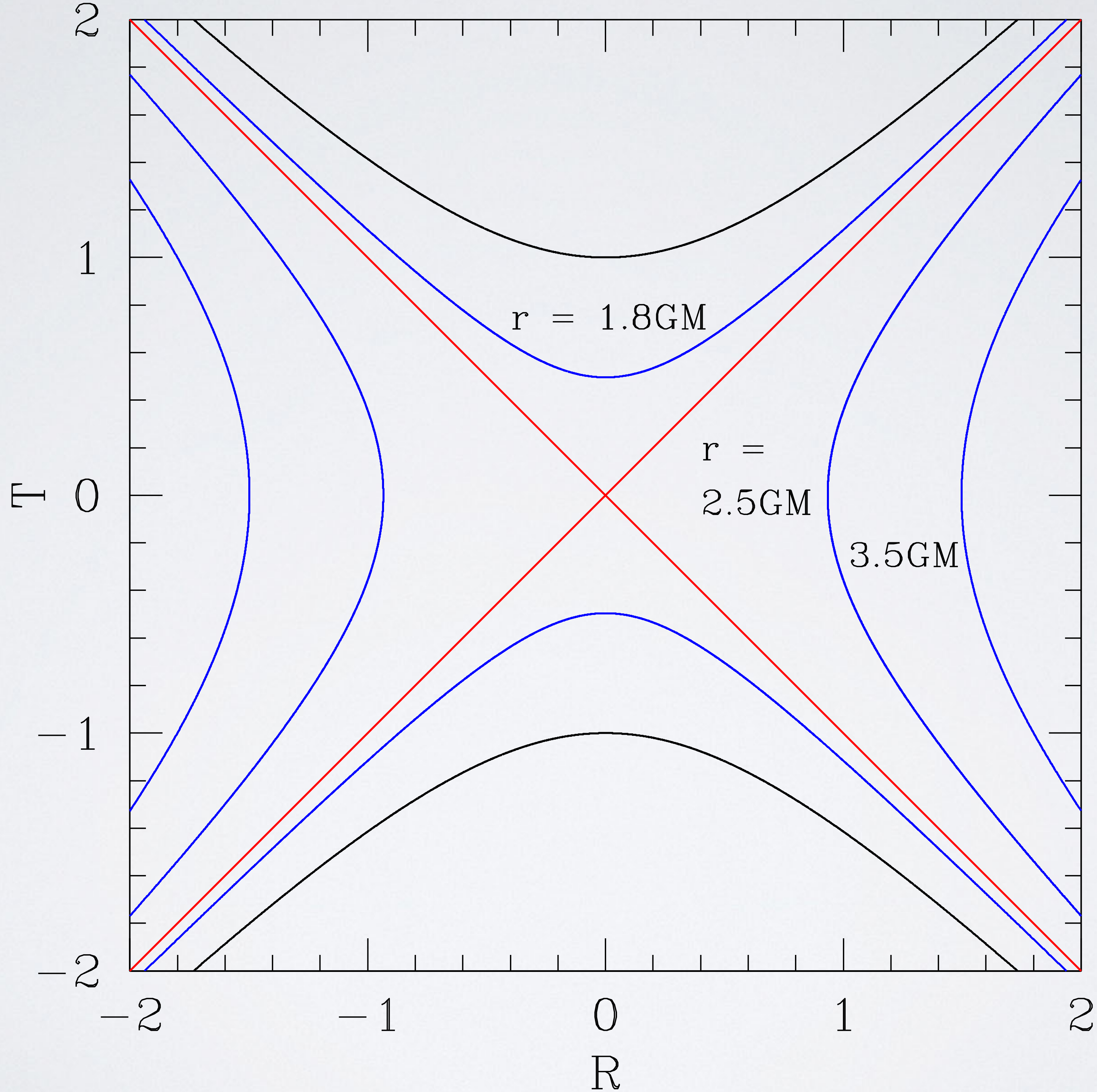

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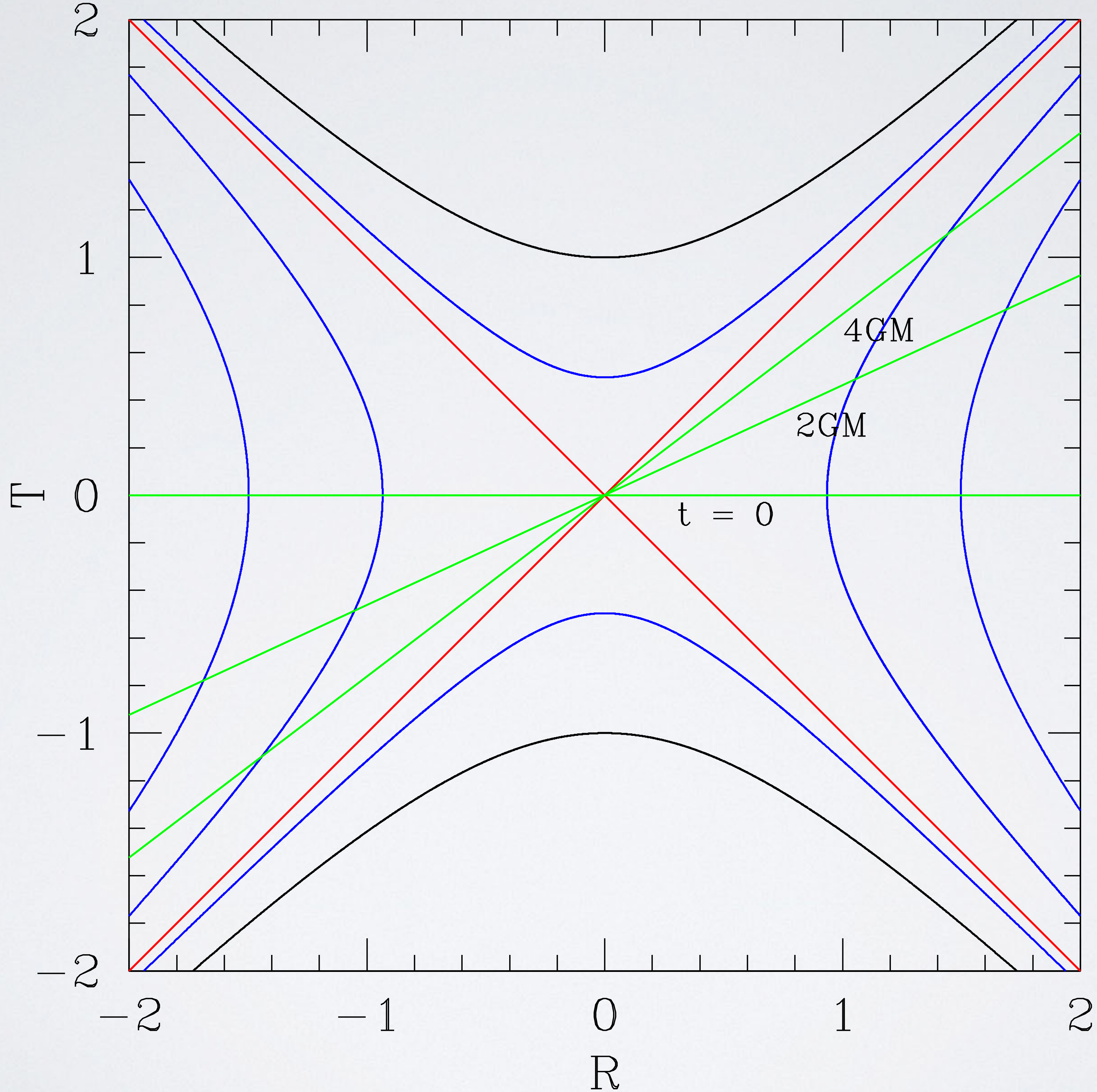

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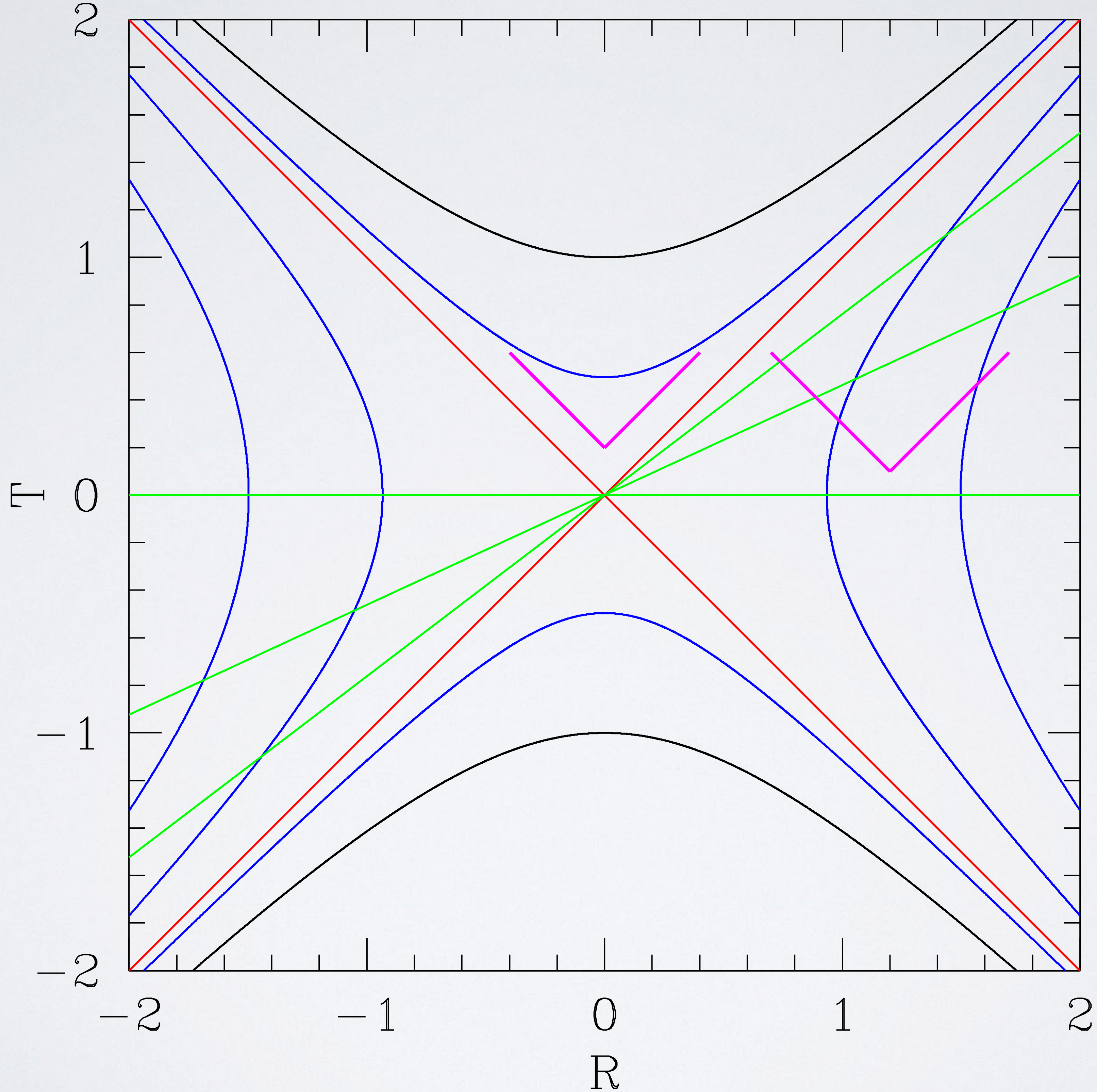

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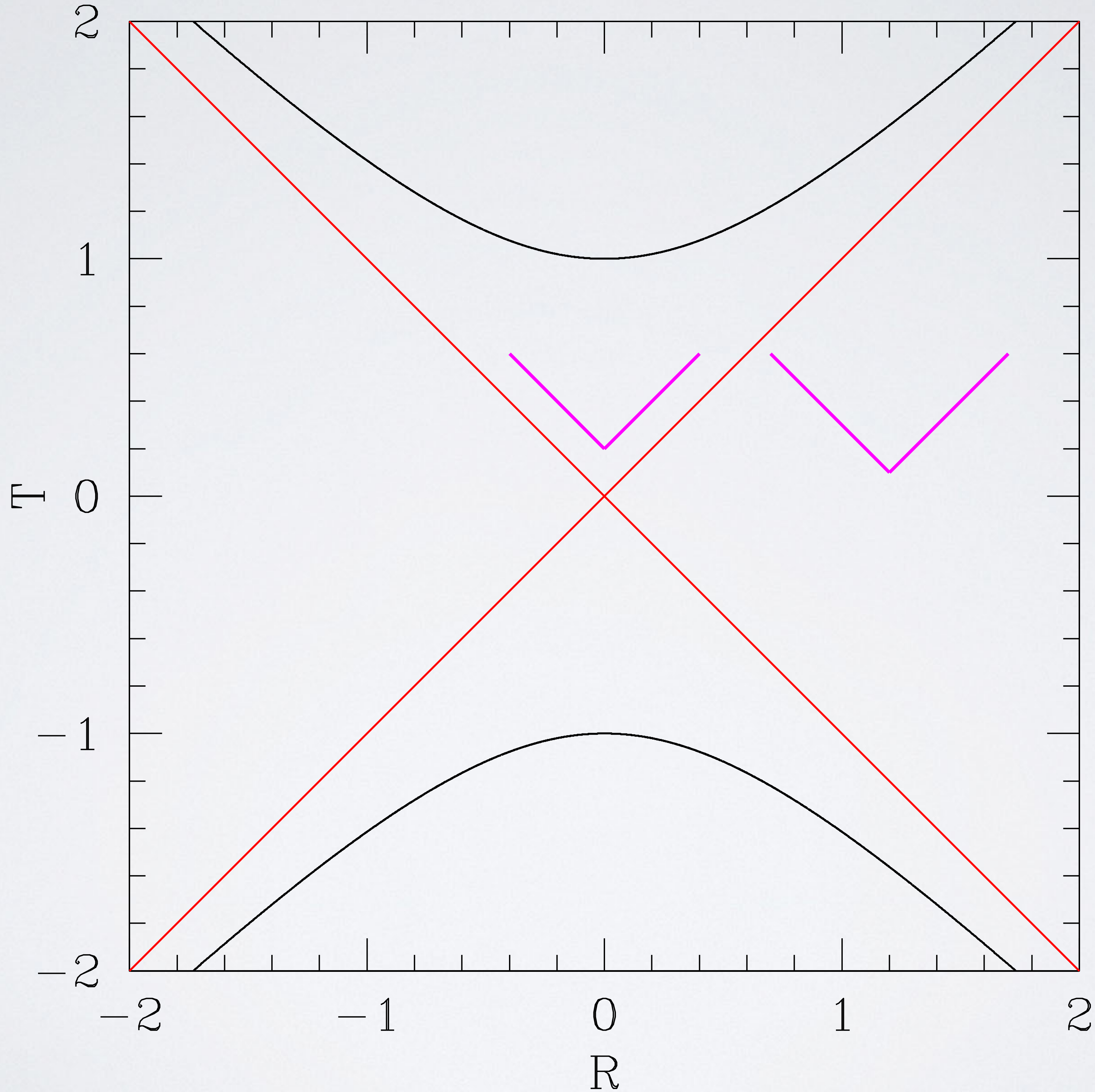

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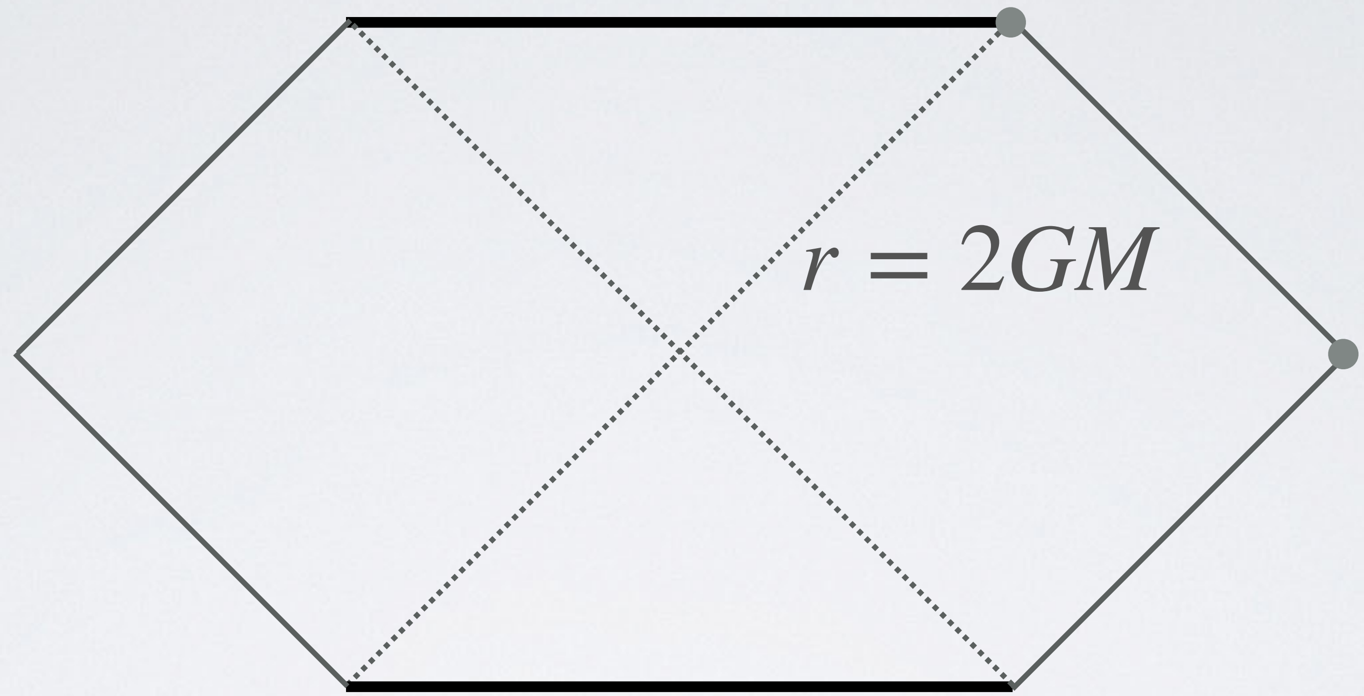

The point $(\psi, \xi) = (\pi/2, \pi/2)$ is the limit of $t \to \infty$ for r finite: "Time-like infinity." Represents the asymptotic future of time-like trajectories on this figure. The point $(\pi/2, \pi)$ represents $r \to \infty$ for finite t: "Space-like infinity." This is where space-like surfaces accumulate on this figure.

The line between them is "null" or "light-like" infinity: This is where light-like trajectories asymptotically end in this figure.

8.962 LECTURE 22

Spacetime of a rotating black hole:

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right)dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$
$$+\frac{\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]}{\rho^2}\sin^2\theta\, d\phi^2 - \frac{4GMar\sin^2\theta}{\rho^2}dt\,d\phi$$

Here we've introduced $a \equiv |\mathbf{S}|/GM$, and the quantities

$$\Delta = r^2 - 2GMr + a^2\ , \quad \rho^2 = r^2 + a^2\cos^2\theta$$

The solution was originally discovered by mathematician Roy Kerr in 1963, and was put into these coordinates by Boyer and Lindquist in 1967 (links will be posted to the 8.962 page).