Summary :

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right) dr^2 + r^2 d\Omega^2$$

is a black hole.

Vacuum everywhere ... except for singular field equations at $r=0$.

Bad coordinates at $r = 2GM$: "Surface of infinite redshift"

"Event horizon": Once in, you never get out. All physical trajectories hit the singularity at $r=0$.

Other black holes:

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right) dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Comes from $T^{\mu}{}_{\nu} = \mathrm{diag}\left[-\frac{Q^2}{8\pi r^4}, -\frac{Q^2}{8\pi r^4}, \frac{Q^2}{8\pi r^4}, \frac{Q^2}{8\pi r^4}\right]$

$\rightarrow$ Black hole metric with a Coulomb electric field!

Horizon is at root of function:

$$r_{horiz} = G\left[M + \sqrt{M^2 - Q^2}\right]$$

Notice: if $|Q| > M$, no horizon! Still a singularity at $r=0$ - "naked".

How do we know that this where the event horizon is located? In general, we don't! Most generally, need to know the entire future null history of the spacetime. Then, event horizon is the null surface which divides regions that allows light rays to reach infinity from those which do not.

If the spacetime is stationary, then we get to chose a clever radial coordinate that makes things easier. Choose r such that it asymptotes to spherical r coordinate at large radius, and such that $r =$ constant surfaces represent timelike worldtubes.

These worldtubes will be intersected by an outgoing light ray! If the spacetime has an event horizon, a well chosen r coordinate will become null at some radius r_H. The diagnostic of this:

$$(\omega^r)_\mu = \partial_\mu r$$

has a norm that goes to zero:

$$g^{\mu\nu} \partial_\mu r \, \partial_\nu r = \cancel{} = 0$$

$$\rightarrow \quad g^{rr}(r_H) = 0.$$

If coordinates are such that $g^{rr}(r_H) = 0$ for all time and all angles, then $r_H =$ event horizon.

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

$$+ \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2$$

$$- \frac{4GMar\sin^2\theta}{\rho^2} dt\, d\phi$$

where $a = |\vec{S}|/M$ $\quad\vec{S}$ = black hole spin

$\Delta = r^2 - 2GMr + a^2$, $\quad \rho^2 = r^2 + a^2\cos^2\theta$

"Kerr black hole in Boyer-Lindquist coordinates"

This is also an exact vacuum solution. Example derivation: Chandrasekhar, "The Mathematical Theory of Black Holes," pp 273-292

Horizon at $\Delta = 0$: $r_H = GM + \sqrt{(GM)^2 - a^2}$

(Note: require $|a| \leq GM$, else naked singularities.)

Coordinates designed to reduce to Schwartschild as $a \to 0$.

Noteworthy features:

1. <u>NOT</u> spherically symmetric! Cannot find a radial coordinate such that $g_{\phi\phi} = \sin^2\theta\, g_{\theta\theta}$.

2. Connection between t & ϕ - $g_{t\phi} = g_{\phi t} = -\frac{2GMar\sin^2\theta}{\rho^2}$

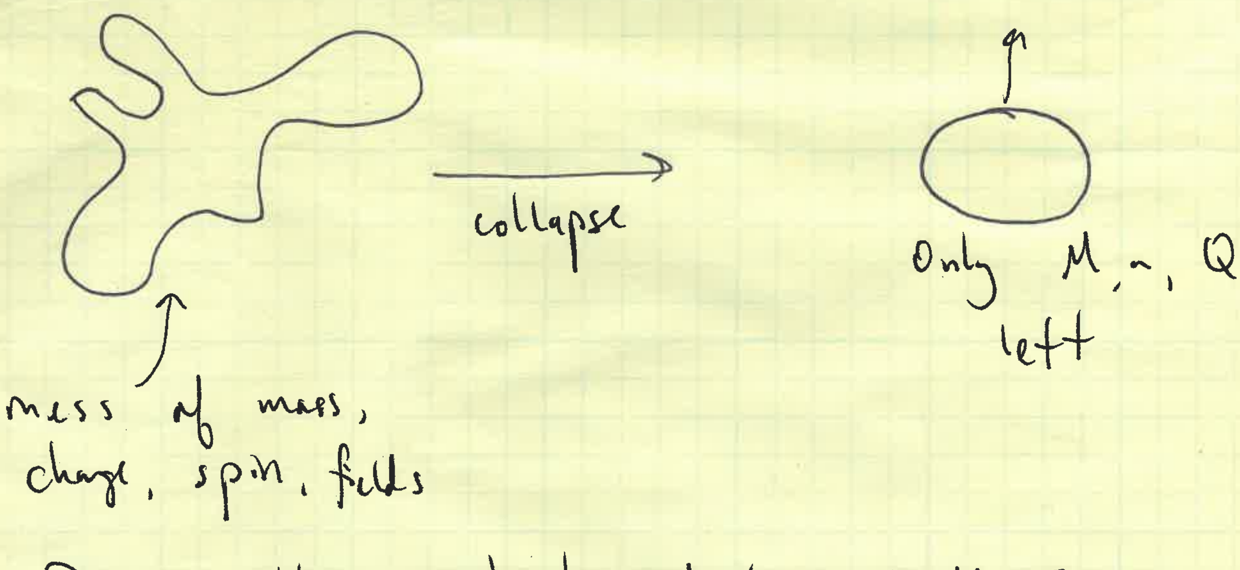
Causes "frame dragging": a geodesic will tend to wrap around in ϕ, parallel to the hole's spin.

Charged, rotating solution also exists: "Kerr - Newman"

Remarkable Theorem: The ONLY stationary spacetimes in 3+1 dimensions with event horizons are the Kerr - Newman black holes, parametrized by mass, spin, and charge.

"stationary" means time independent. In astrophysical contexts, net charge is rapidly neutralized by environmental plasma → Kerr is most relevant.

Known as "No-hair" theorem. Enforcement is interesting: Consider collapse of some complicated object to a black hole



During collapse, strongly radiates: E&M, GWs ... backreaction of radiation removes all structure except M, a, Q.

"Price's theorem": Everything that can be radiated IS radiated.

Motion in black hole spacetimes

Naive approach: Grind out all connection coefficients, study geodesic equation.

See Carroll Eq (5.53) for result!

Not wrong, but not useful. Better to examine symmetry and see what symmetries allow us to simplify things.

Schwarzschild:

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

- Spherical: I can always rotate coordinates such that they lie in the "equatorial" plane, $\theta = \pi/2$. Also guarantees that orbit remains confined to its initial plane: No way to make a torque that changes the plane's orientation.
- $\partial_t g_{\mu\nu} = 0$. Means p_t = constant → means there is a time like Killing vector → means orbits have a notion of conserved energy, $E = -p_t$.
- $\partial_\phi g_{\mu\nu} = 0$. Means p_ϕ = constant → axial Killing vector → conserved angular momentum, $L_z = p_\phi$.

Holds for Reissner-Nordstrom as well! 2ND points hold for Kerr.

4-momentum of a body moving in Schwarzschild:

$$p^\mu \doteq m\left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$$

rest mass of body

choice of orbital plane.

Look at components that are constant:

$$p_\mu = g_{\mu\nu} p^\nu$$

$$\rightarrow \quad p_t = -m\left(1 - \frac{2GM}{r}\right)\frac{dt}{d\tau} \equiv -E$$

Defines conserved energy of the orbit.

$$\rightarrow \quad p_\phi = m r^2 \sin^2\theta \, \frac{d\phi}{d\tau} \equiv L_z$$

$\hookrightarrow$ choose $\theta = \pi/2$

$$= m r^2 \frac{d\phi}{d\tau} \equiv L.$$

Defines conserved angular momentum of orbit.

Now, use $g_{\mu\nu} p^\mu p^\nu = -m^2$:

$$-m^2\left(1 - \frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + m^2\left(1 - \frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + m^2 r^2\left(\frac{d\phi}{d\tau}\right)^2 = -m^2$$

Replace: $\dfrac{dt}{d\tau} = +\dfrac{E}{m}\left(1 - \dfrac{2GM}{r}\right)^{-1} \equiv \hat{E}\left(1 - \dfrac{2GM}{r}\right)^{-1}$

$$\frac{d\phi}{d\tau} = \frac{L}{m r^2} \equiv \frac{\hat{L}}{r^2}$$

& multiply everything by $1 - 2GM/r$:

$$\rightarrow \quad -m^2 \hat{E}^2 + m^2 \left(\frac{dr}{d\tau}\right)^2 + m^2 \left(1 - \frac{2GM}{r}\right) \frac{\hat{L}^2}{r^2} = -m^2 \left(1 - \frac{2GM}{r}\right)$$

Rearrange:

$$\boxed{\begin{aligned} \left(\frac{dr}{d\tau}\right)^2 &= \hat{E}^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right) \\ &\equiv \hat{E}^2 - V_{eff}(\hat{L}, r) \end{aligned}}$$

Studying trajectories of bodies near Schw. black holes boils down to a simple recipe:

1. Pick energy per unit mass $\hat{E}$ and angular momentum per unit mass $\hat{L}$.

2. Pick initial position (r, ϕ).

3. Integrate $\left(\frac{dr}{d\tau}\right)^2 = \hat{E}^2 - V_{eff}(\hat{L}, r)$

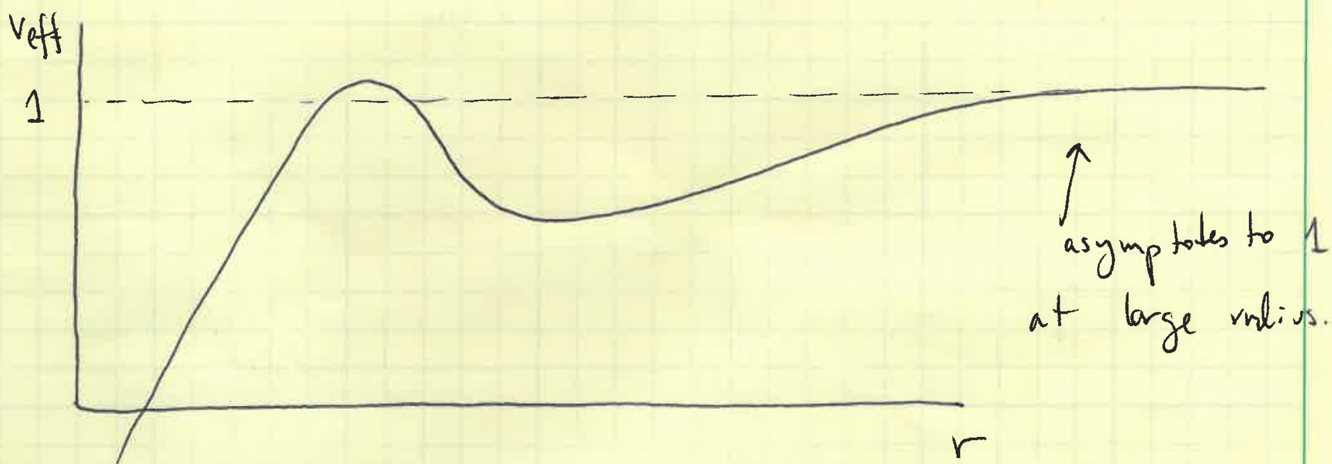
$$\frac{d\phi}{d\tau} = \frac{\hat{L}}{r^2}$$

$$\frac{dt}{d\tau} = \frac{\hat{E}}{1 - 2GM/r}$$

All of the interesting behavior is bound up in the function

$$V_{eff} = \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right)$$

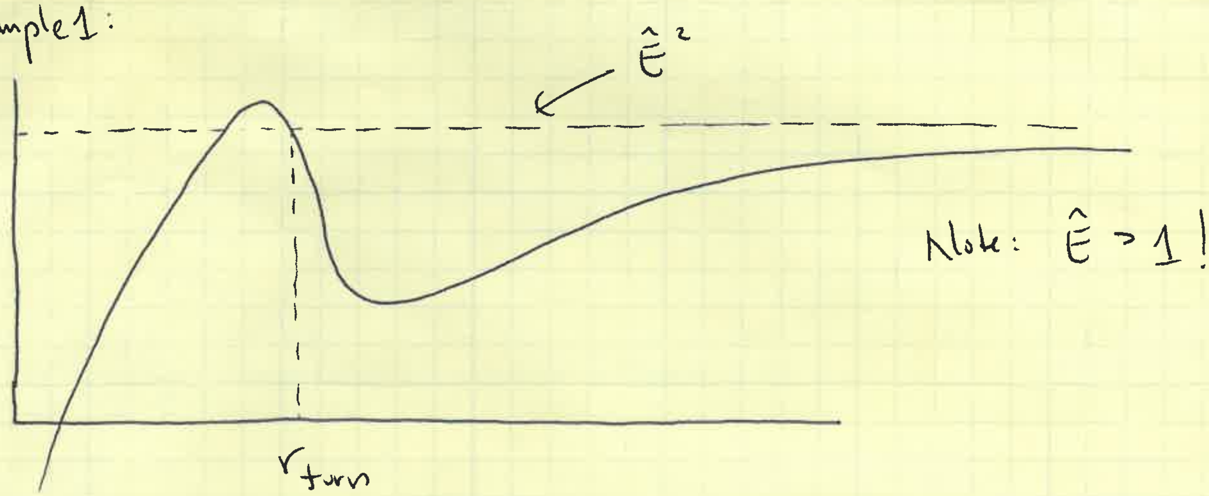
Typical shape of "potential" given $\hat{L}$:



$\hat{E}^2$ has the same units as V_{eff}. Can plot them together to understand radial motion.

$$\left(\frac{dr}{d\tau}\right)^2 = \hat{E}^2 - V_{eff}(r, \hat{L})$$

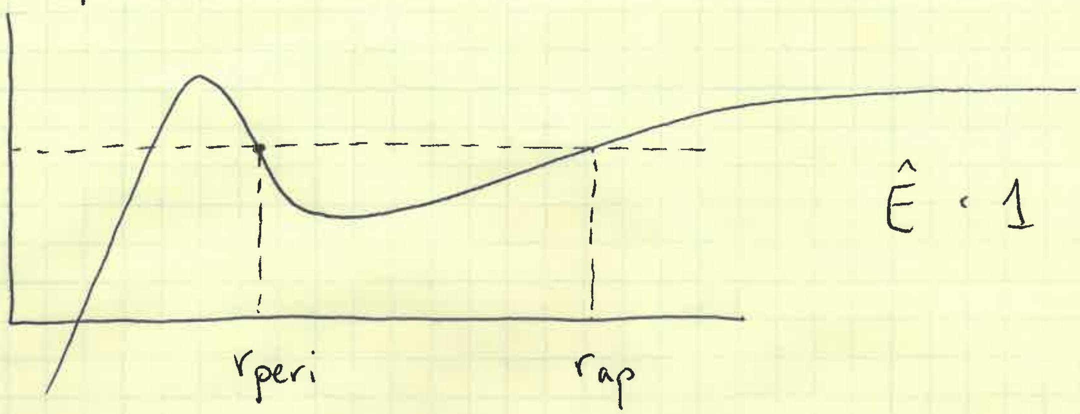
Example 1:



For this $\hat{E}$, body comes in from infinity, turns around at r_{turn} [$dr/d\tau = 0$ at turning point: $\hat{E} = \sqrt{V_{eff}(r_{Turn})}$], goes back out to infinity.

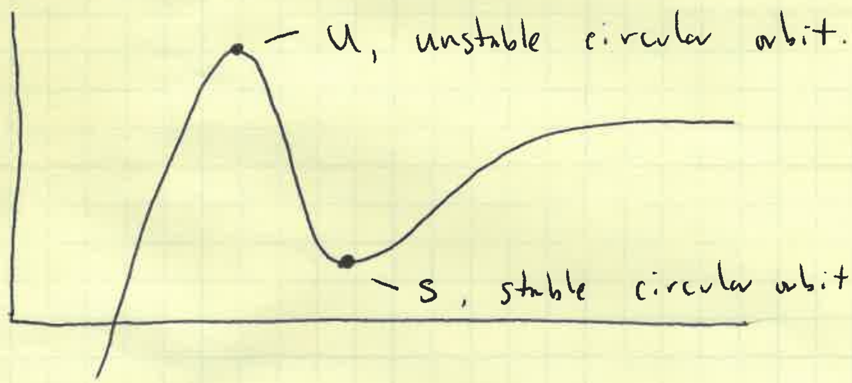
Relativistic generalization of "hyperbolic orbit".

Example 2:



Orbits only have $\hat{E}^2 > V_{eff}$ in the range $r_p \le r \le r_a$; otherwise, $(dr/d\tau)^2 < 0$.

These orbits give relativistic generalization of eccentric orbits. Suggests that for each potential there should be circular orbits as well:



Conditions for circularity:

$$\frac{dr}{d\tau} = 0 \rightarrow \hat{E} = \sqrt{V_{eff}}$$

$$V_{eff} = \text{min or max} \rightarrow \frac{\partial V_{eff}}{\partial r} = 0$$

$$\frac{\partial V_{eff}}{\partial r} = 0 \quad \rightarrow \quad 2\hat{L}^2 (r - 3GM) = 2GMr^2$$

$$\rightarrow \boxed{\hat{L} = \pm \sqrt{\frac{GMr}{1 - 3GM/r}}}$$

Asymptotes to $\hat{L} = \pm\sqrt{GMr}$ for large r → same as Newtonian value.

Enforce $\hat{E} = \sqrt{V_{eff}}$ for this value of $\hat{L}$:

$$\boxed{\hat{E} = \frac{1 - 2GM/r}{\sqrt{1 - 3GM/r}}}$$

Notice. $\hat{E} < 1$. Intuitively, can regard

$$\hat{E} = \frac{E}{m} = \frac{(m + E_{kinetic} + E_{potential})}{m}$$

Bound orbits have $|E_{pot}| > E_{kin}$; and $E_{pot} < 0$.
Hence, $\hat{E} < 1$. Note that

$$\hat{E} \rightarrow 1 - \frac{GM}{2r} \quad \text{as} \quad r \rightarrow \infty \quad -$$

Newtonian value for circular orbits.

Useful quantities: the angular velocity of circular orbits as seen by distant observers.

$$\Omega \equiv \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{\hat{L}/r^2}{\hat{E}}\left(1 - \frac{2GM}{r}\right)$$

$$= \frac{1}{r^2} \cdot \sqrt{GMr} = \sqrt{\frac{GM}{r^3}} \quad \rightarrow \quad \text{same as Newtonian value!}$$

Orbits are stable if $\partial^2 V/\partial r^2 > 0$, unstable if $\partial^2 V/\partial r^2 < 0$. What if stable & unstable orbits coincide? Only have a <u>marginally</u> stable orbit:

$$\frac{\partial^2 V}{\partial r^2} = 0 = \frac{6\hat{L}^2(r-4GM) - 4GMr^2}{r^5}$$

$$= \frac{2GM(r-6GM)}{r^3(r-3GM)}$$ (subst. our solution for $\hat{L}$)

$$\longrightarrow \boxed{r = 6GM}$$

<u>No</u> stable circular orbit exists inside $r = 6GM$!
Very non-Newtonian behavior.