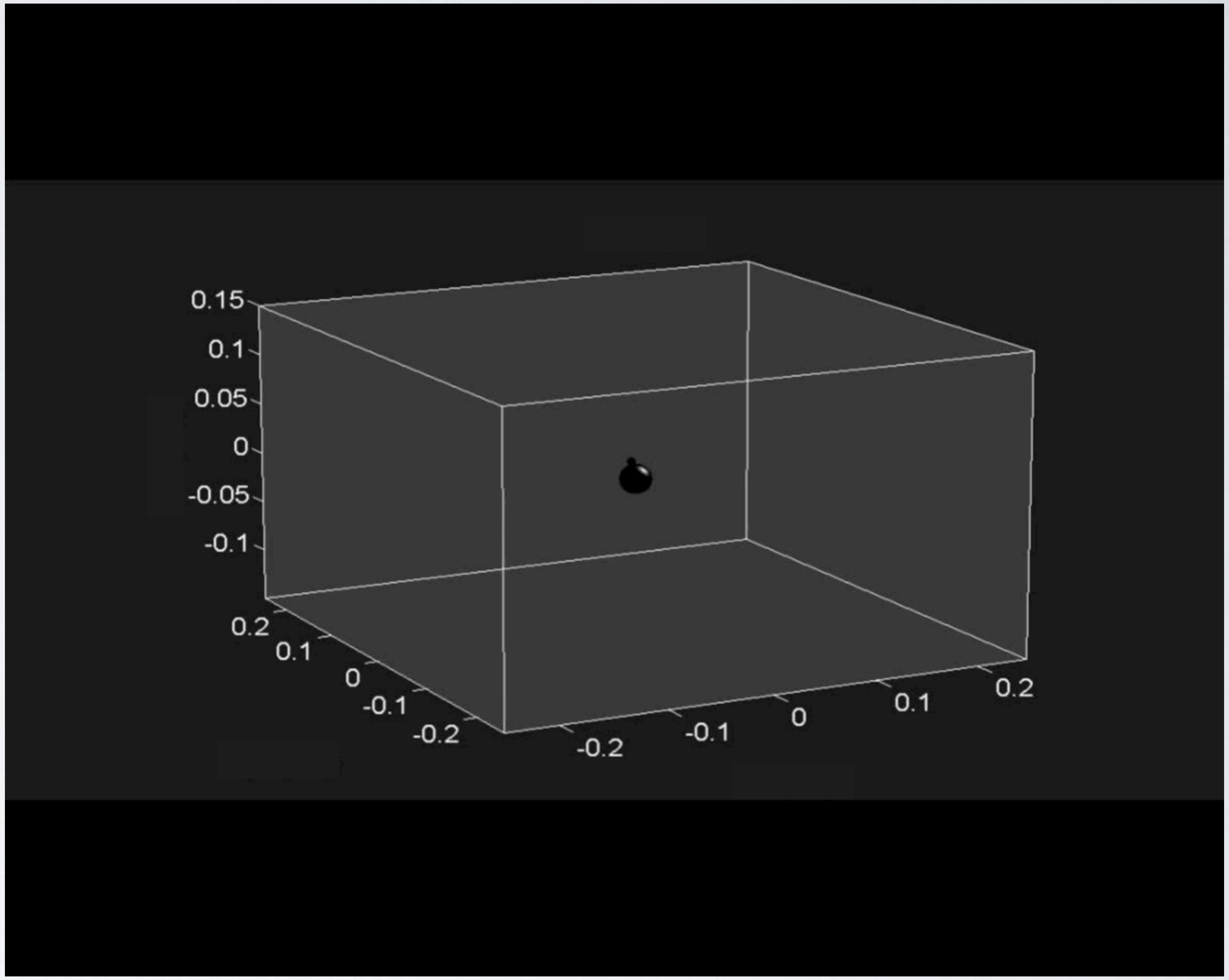
8.962 LECTURE 23

More black holes
Black hole orbits



Small body orbiting a rotating black hole (video produced by former UROP student Peter Reinhardt)

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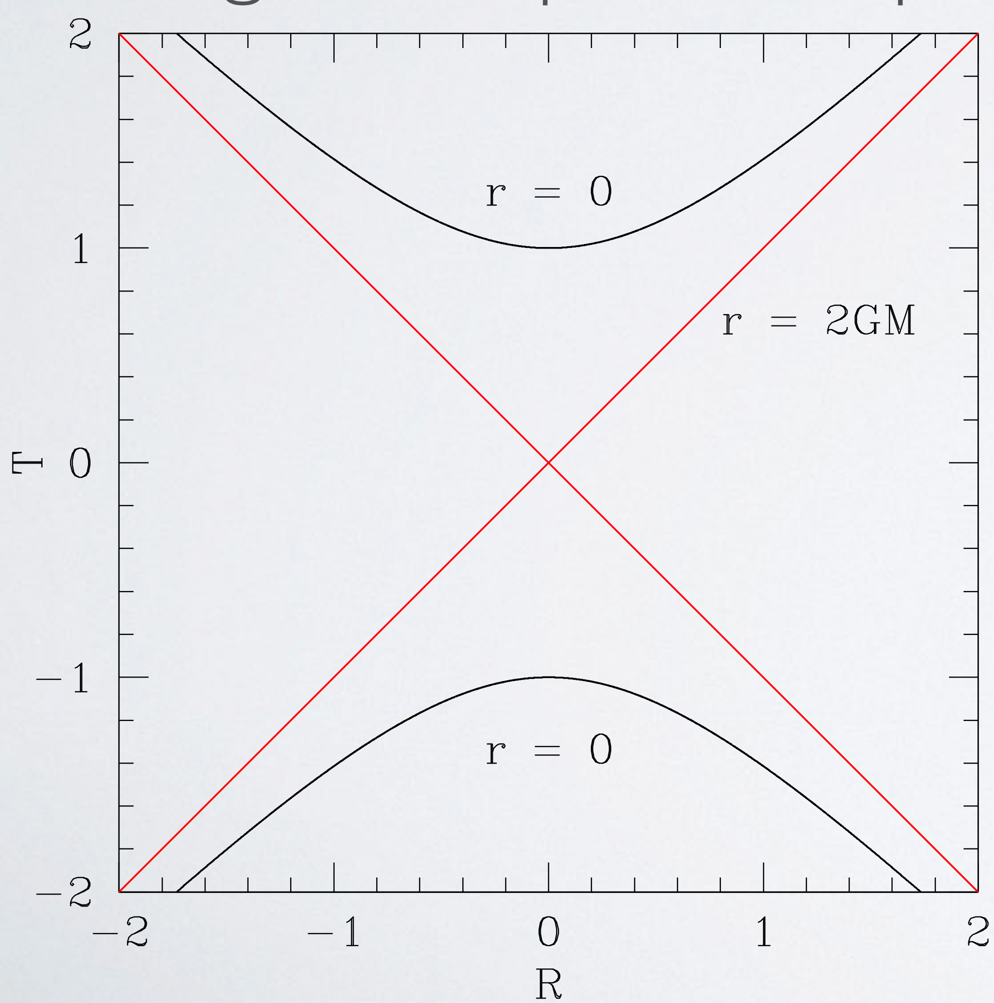
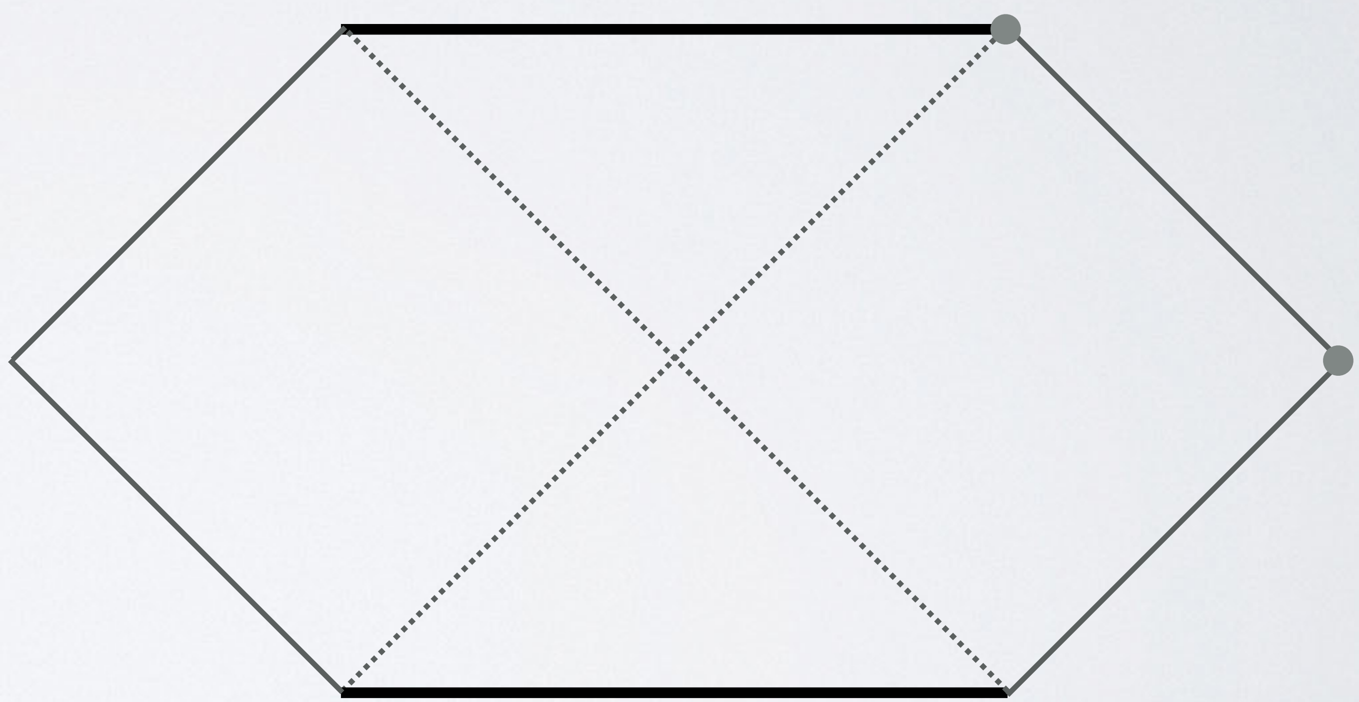
Recap: Carefully examined the vacuum Schwarzschild spacetime,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - 2GM/r} + r^2 d\Omega^2$$

Particularly important: methods developing its *causal* structure, showing which parts of spacetime are in causal contact:





Penrose diagram

Kruskal-Szekeres coordinates

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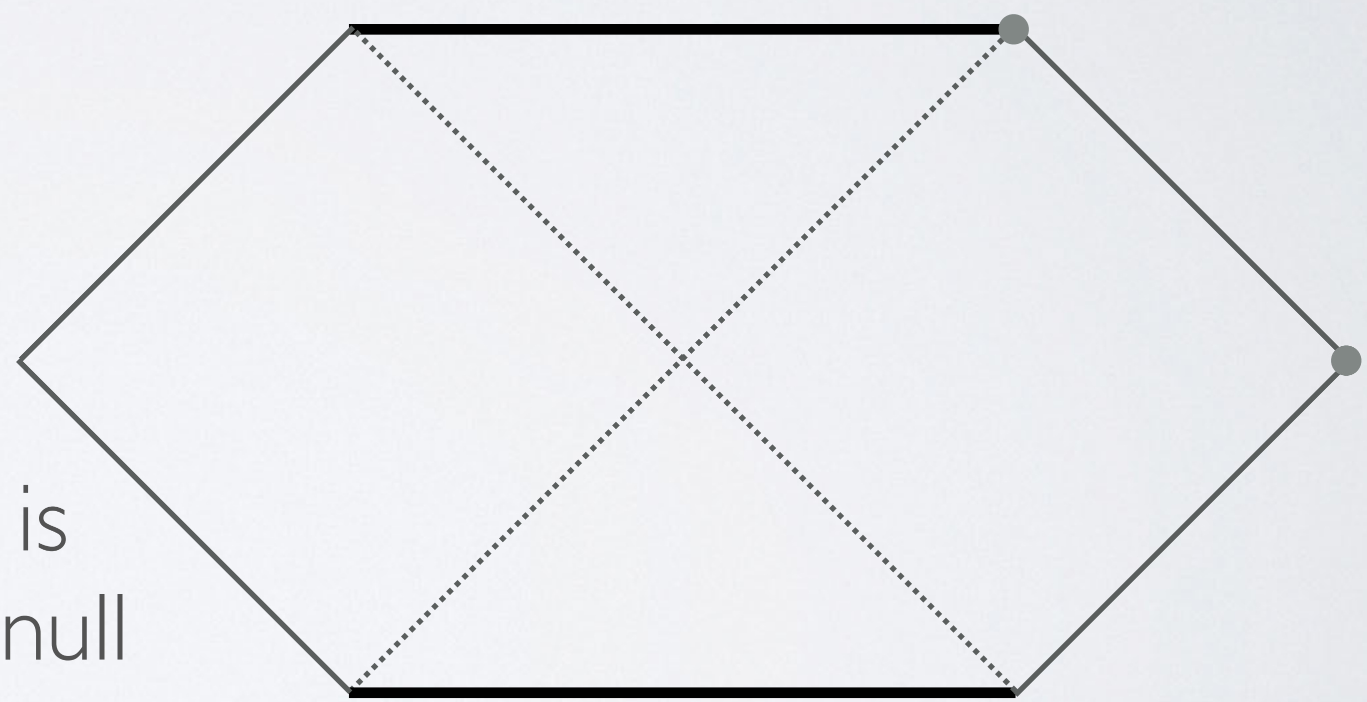

Key things we learn:
* Horizon at $r = 2GM$ is a null surface; time t approaches infinity as we reach it.
* Curvature singularity at $r = 0$ is in the future of all timelike and null trajectories that cross $r = 2GM$.

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Spacetime of a charged black hole ("Reissner-Nordstrom"):

$$ds^2 = -\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)dt^2 + \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Not too difficult to compute this solution: Just repeat the exercise we did for Schwarzschild vacuum, but insert the stress-energy tensor of a Coulomb field centered on $r = 0$.

Curvature again singular at $r = 0$:

$$I = \frac{48G^2M^2}{r^6} - \frac{96G^2MQ^2}{r^7} + \frac{56G^2Q^4}{r^8}$$

Horizon located at root of function which appears in g_{tt} and g_{rr}:

$$r_{\rm H} = GM + \sqrt{(GM)^2 - GQ^2}$$

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Spacetime of a rotating black hole:

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$
$$+ \frac{\left[(r^2 + a^2)^2 - a^2 \Delta \sin^2\theta\right]}{\rho^2} \sin^2\theta \, d\phi^2 - \frac{4GMar\sin^2\theta}{\rho^2} dt\, d\phi$$

Here we've introduced $a \equiv |\mathbf{S}|/GM$, and the quantities

$$\Delta = r^2 - 2GMr + a^2 \,, \quad \rho^2 = r^2 + a^2\cos^2\theta$$

Horizon at root $\Delta = 0$: $r_{\mathrm{H}} = GM + \sqrt{(GM)^2 - a^2}$.

Singularity at $\rho = 0$:

$$I = \frac{48G^2M^2(r^6 - 15a^2r^4\cos^2\theta + 15a^4r^2\cos^4\theta - a^6\cos^6\theta)}{(r^2 + a^2\cos^2\theta)^6}$$

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Notice the off-diagonal term! The connection between t and ϕ will make an important contribution to the kinematics of bodies moving near a black hole — develops "frame dragging," tendency of inertial bodies to be dragging into moving in the same sense as the hole's rotation.

Intuitively, the rotating black hole (which, as a vacuum solution, is really just rotating spacetime) drags a region of spacetime into co-rotation with it.

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Not spherically symmetric! If it were spherical, then we would have $g_{\phi\phi} = g_{\theta\theta}\sin\theta$; this is only true in the Schwarzschild limit (i.e., nonspinning: $a = 0$).

Good diagnostic of asphericity: Circumference of the event horizon in the equatorial plane

$$C_{\rm eq} = \int_0^{2\pi} \sqrt{g_{\phi\phi}(r = r_{\rm H}, \theta = \pi/2)}\,d\phi = 4\pi GM$$

versus circumference going around the poles:

$$C_{\rm pol} = 2\int_0^{\pi} \sqrt{g_{\theta\theta}(r = r_{\rm H}, \theta)}\,d\theta$$

Polar circumference can be expressed in closed form using elliptic integrals, but numerical values are the key things that matter.

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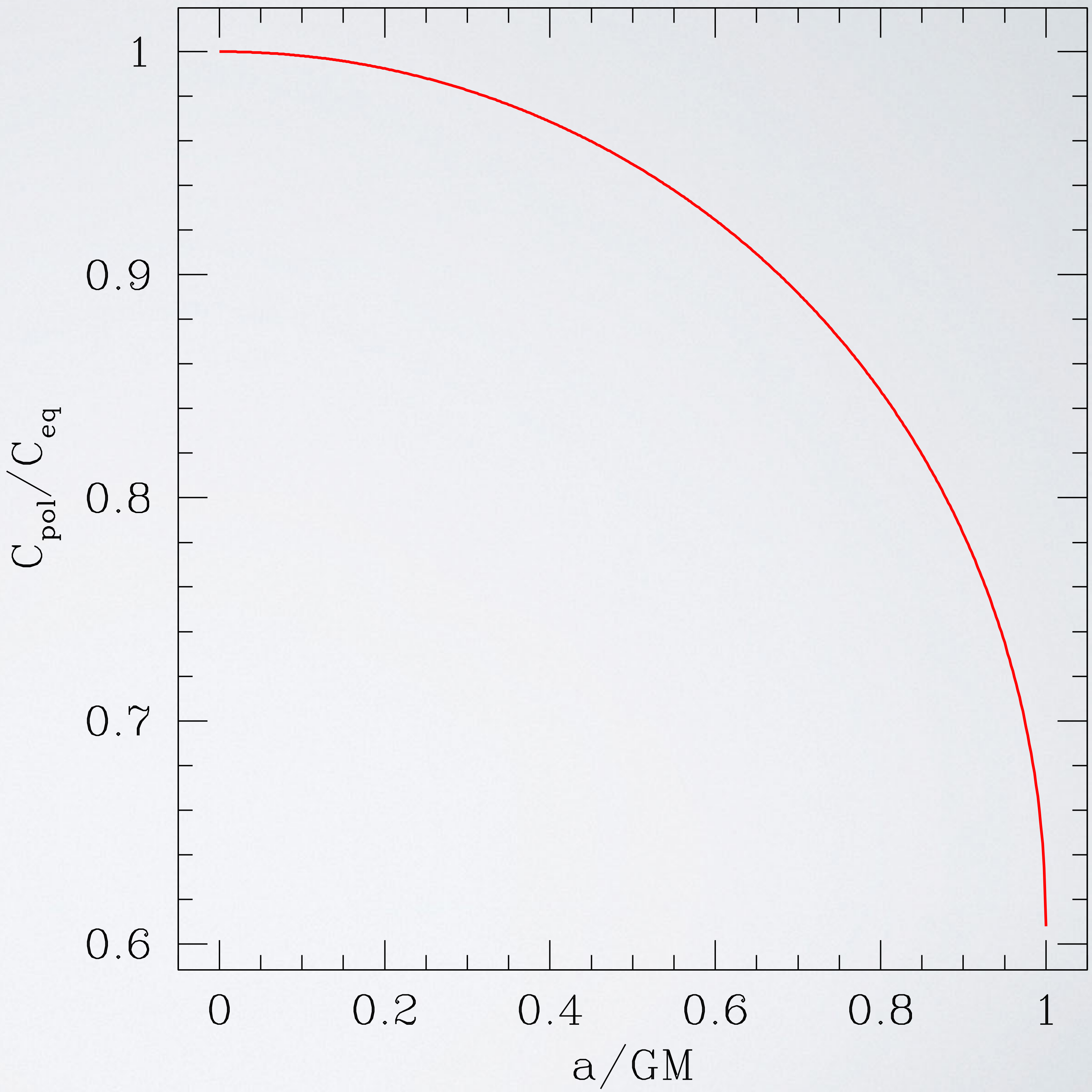
Result: significant difference as the spin parameter approaches 1.

Horizon is defined by a single radial coordinate, but has an oblate geometry … can think of this as centrifugal flattening due to its rotation.



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I have yet to find or develop a straightforward derivation of this spacetime! Not hard to show, given this metric, that it satisfies vacuum Einstein field equations (GRTool.nb tuned to the Kerr metric posted to 8.962 Canvas page).

Original derivation by Kerr sought to categorize metrics that take the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + 2Hk_\alpha k_\beta$$

where H is a scalar field, and k^α describes null vector. Original paper by Roy Kerr (1963) identifies an example of this spacetime as describing a spinning mass (compare exercise on pset 7 of line element). Boyer and Lindquist (1967) put in into a coordinate system that reduced to Schwarzschild when the spin parameter is zero, made it clear that the spacetime has event horizons.

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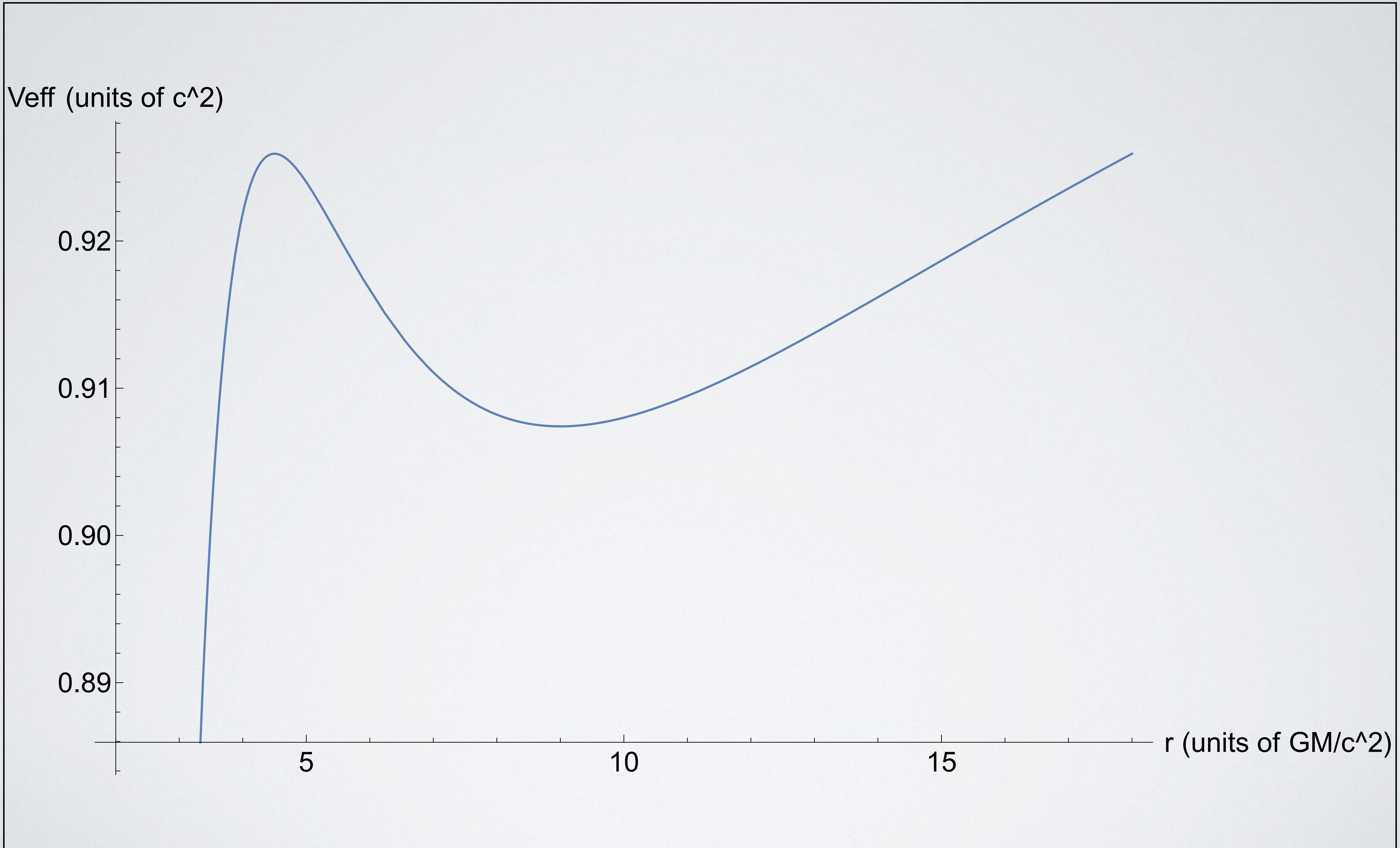

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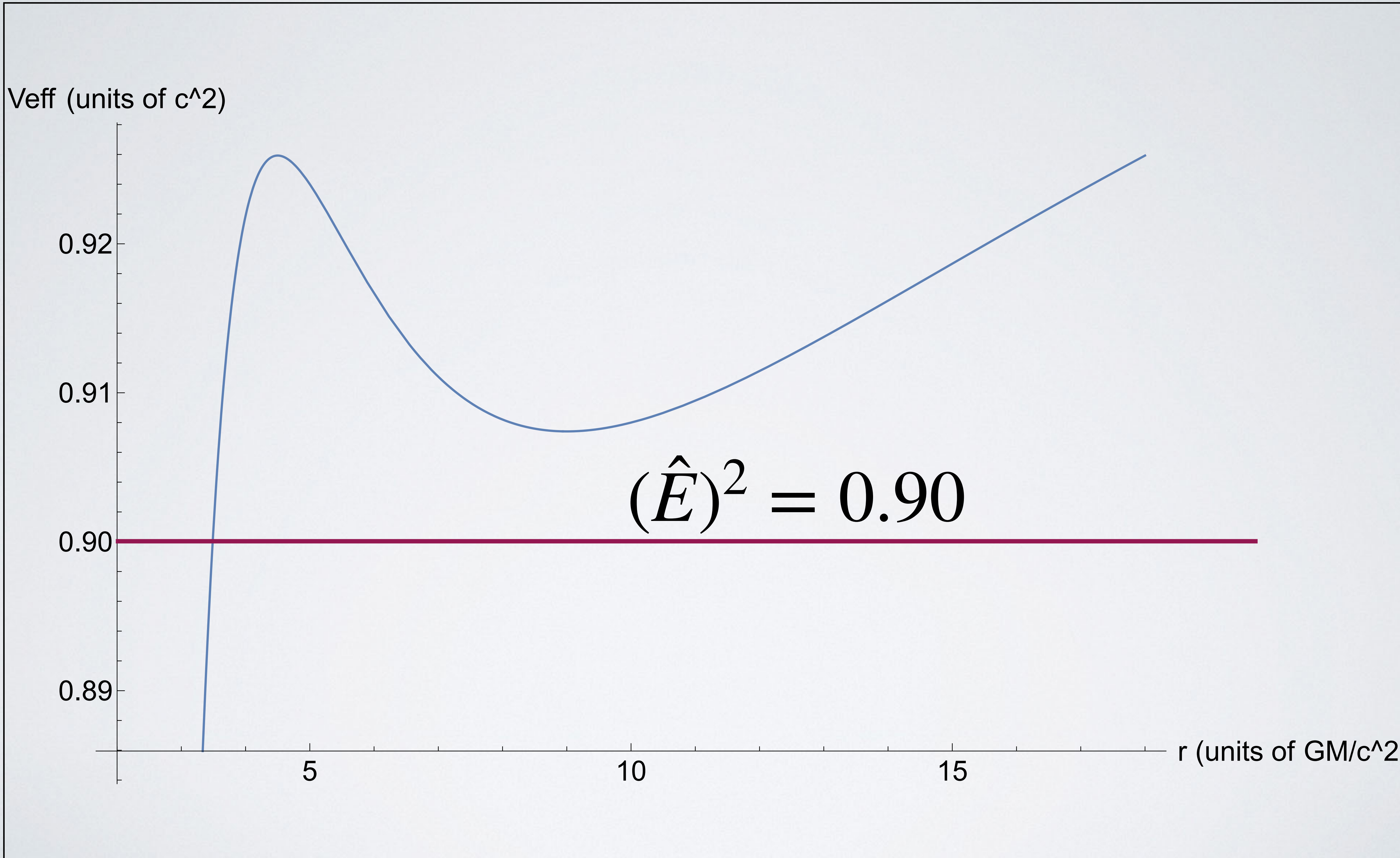

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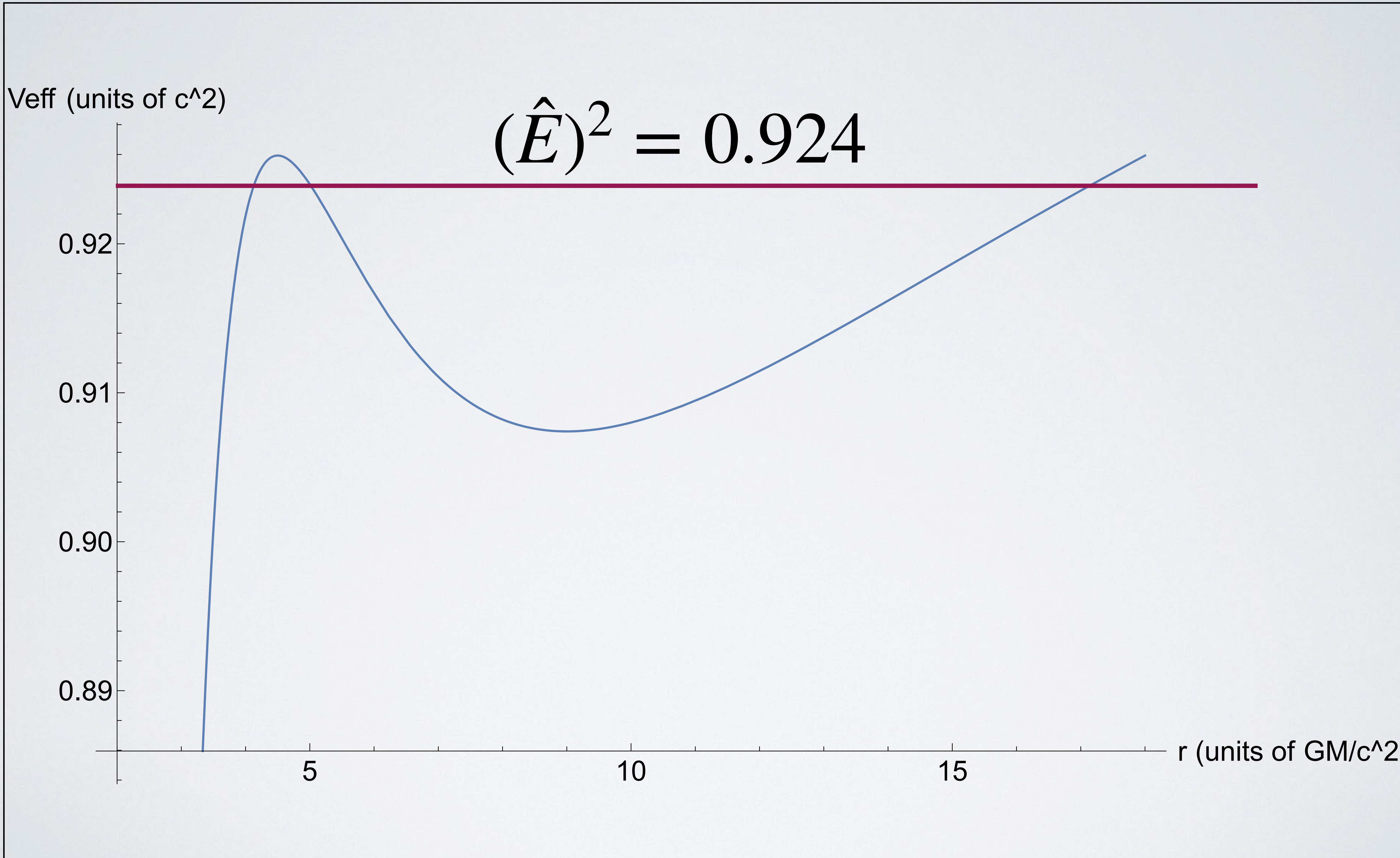

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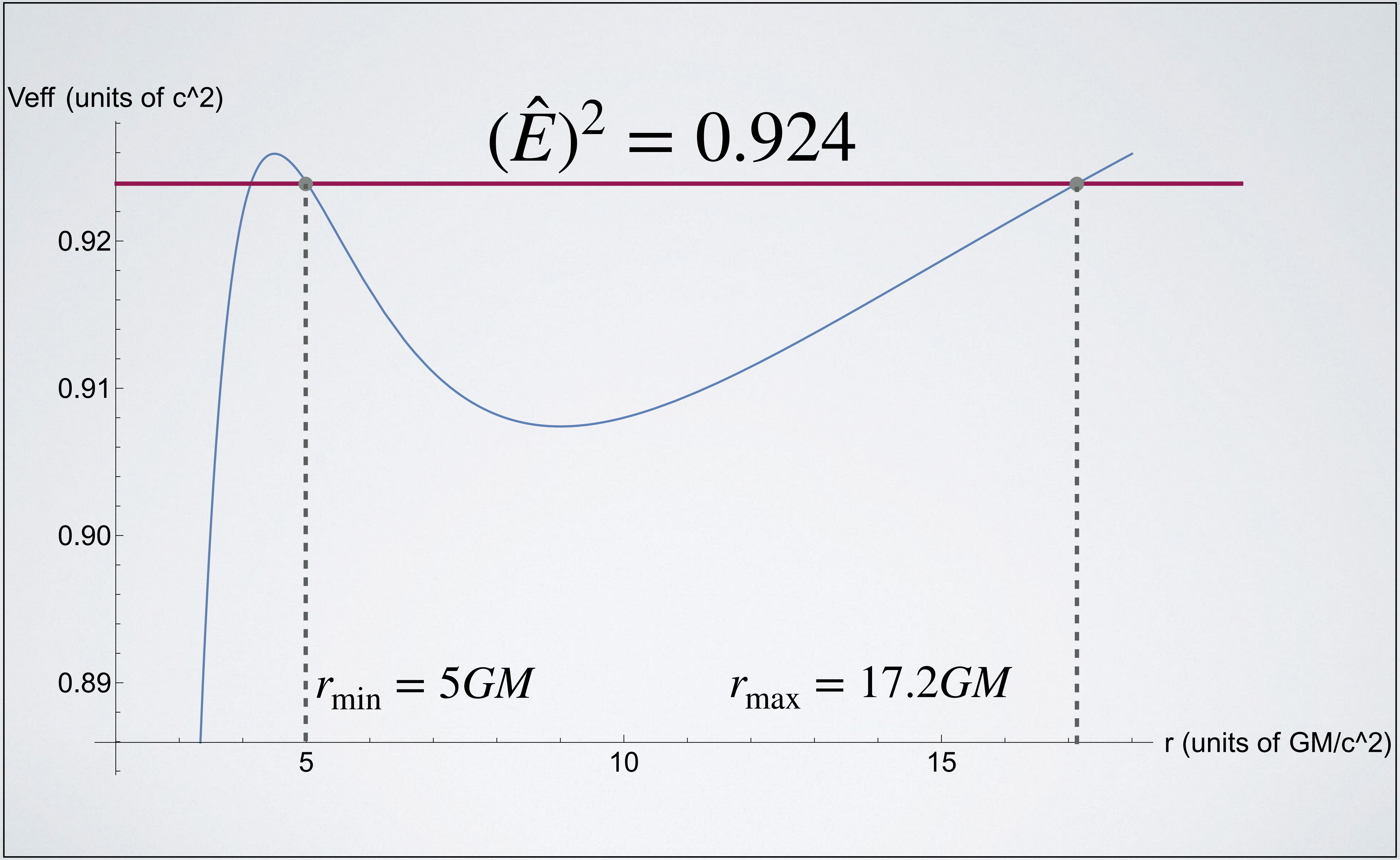

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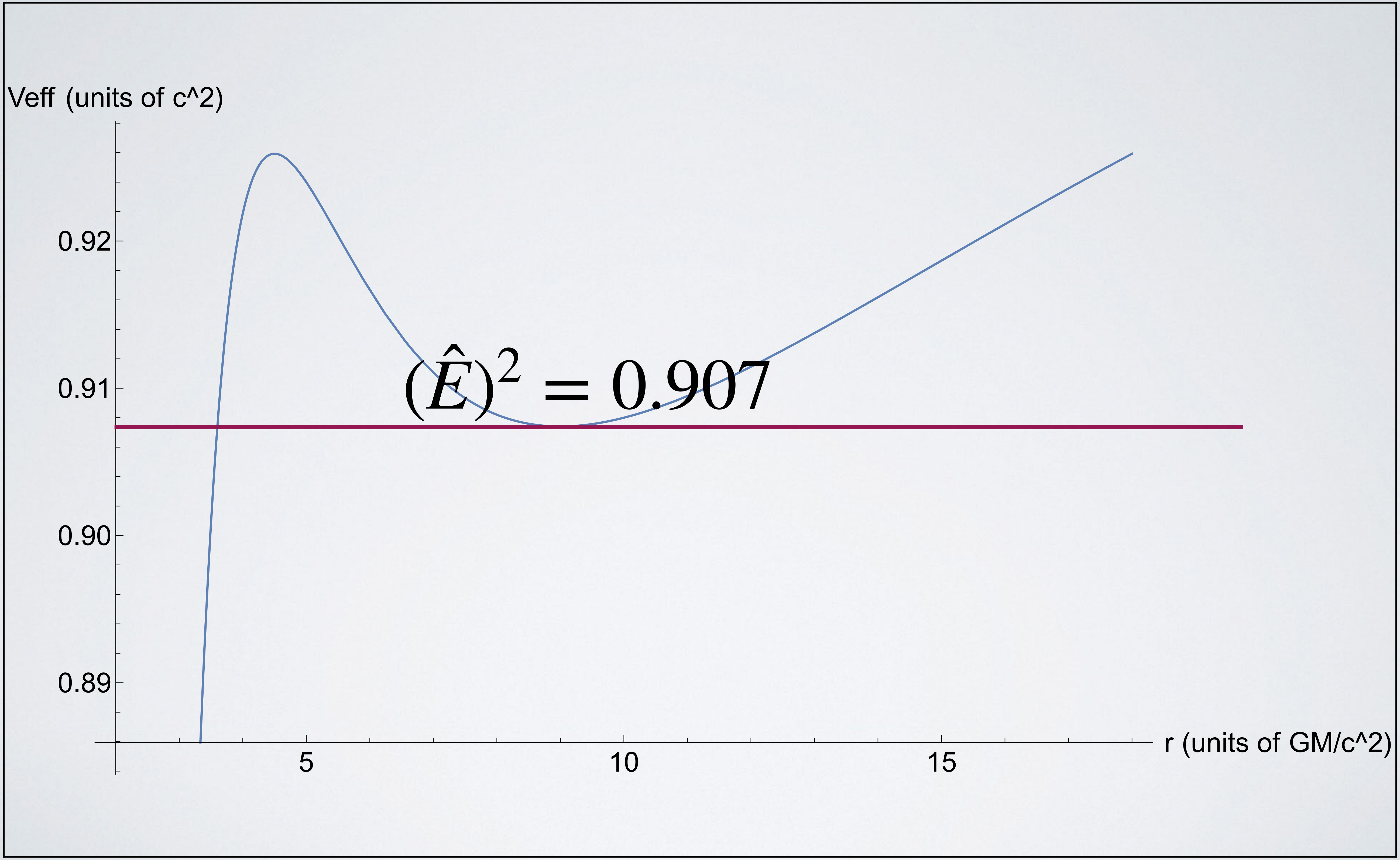

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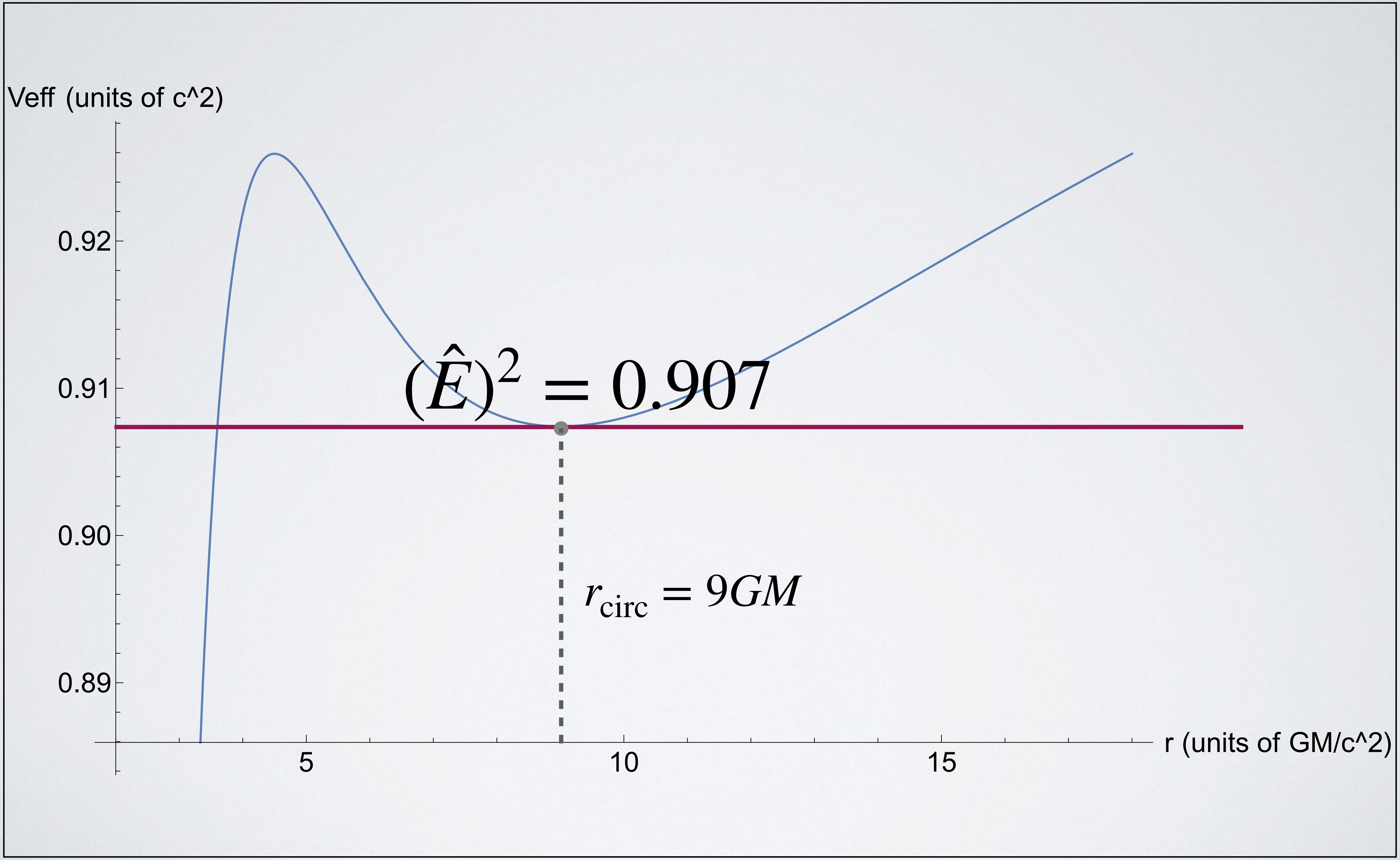

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