"Photon Orbits"

Recall that for null geodesics, we define the affine parameter such that $\vec{p} = d\vec{x}/d\lambda$. Null means $\vec{p}\cdot\vec{p}=0$. Putting all this together leads to a rather different potential describing the motion:

$$\vec{p}\cdot\vec{p}=0 \rightarrow \quad 0 = -\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2$$

Still have conserved energy and angular momentum:

$$E = -p_t = \left(\frac{dt}{d\lambda}\right)\left(1-\frac{2GM}{r}\right)$$

$$L = p_\phi = r^2\frac{d\phi}{d\lambda}$$

These relations can be rewritten to yield equations for the worldlines followed by light in BH spacetime:

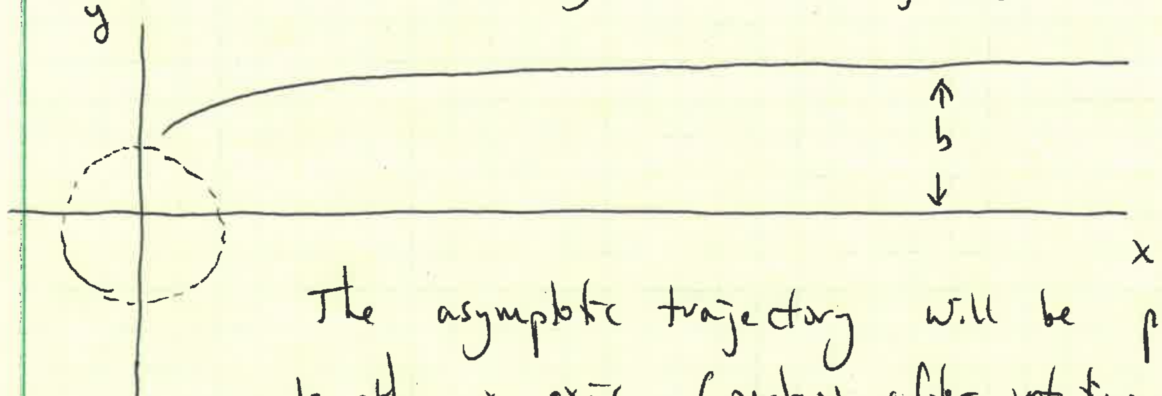
$$\left(\frac{dt}{d\lambda}\right) = \frac{E}{1-2GM/r} \qquad \left(\frac{d\phi}{d\lambda}\right) = \frac{L}{r^2}$$

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - \frac{L^2}{r^2}\left(1-\frac{2GM}{r}\right)$$

PROBLEM: trajectory should not depend on energy. We are working in the "geometric optics limit," and expect gamma rays and radio waves to follow the same paths.

Solution: Redefine the affine parameter, putting $\lambda \to L\lambda$, defining $b \equiv L/E$. Note that b is a length.

To get intuition about what b means, consider light ray moving in the (r, ϕ) plane. Let us define $x = r\cos\phi$, $y = r\sin\phi$. Propagate a light ray from near $r = 2GM$ with non zero angular momentum ... at large radius, it will move on a nearly radial trajectory:



The asymptotic trajectory will be parallel to the x axis (perhaps after rotating coordinates), but offset by coordinate distance b.

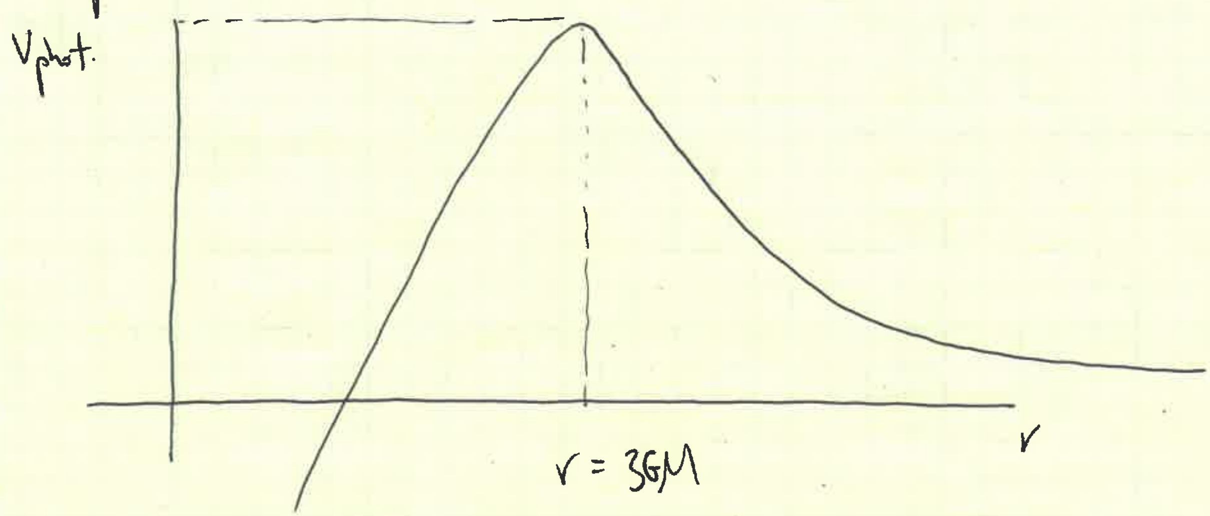
b is exactly like "impact parameter" in Newtonian physics.

Using b, we can rewrite the radial equation of motion as

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2GM}{r}\right)$$

$$= \frac{1}{b^2} - V_{phot}(r)$$

Notice the photon potential doesn't depend on <u>any</u> parameters:



Peak is at $V_{phot} = 1/27G^2M^2$

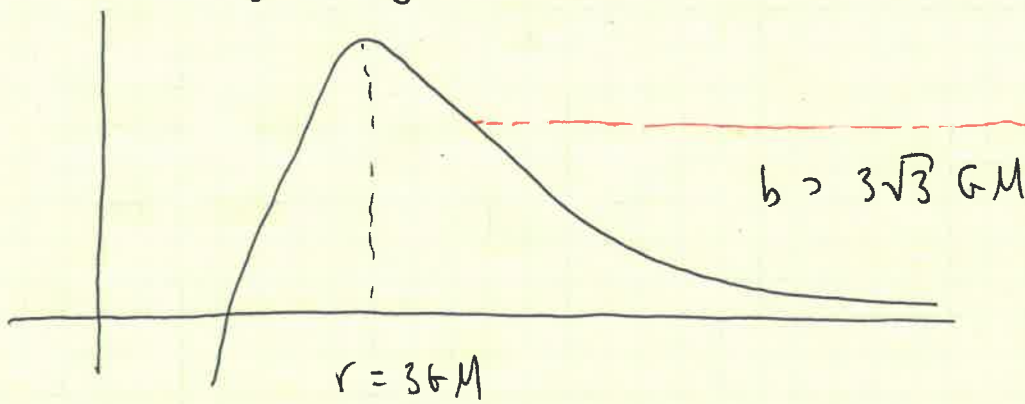
$$= 1/[3\sqrt{3}\,GM]^2$$

Behavior of light moving near BH follows by comparing $1/b^2$ with V_{phot}, esp. the peak.

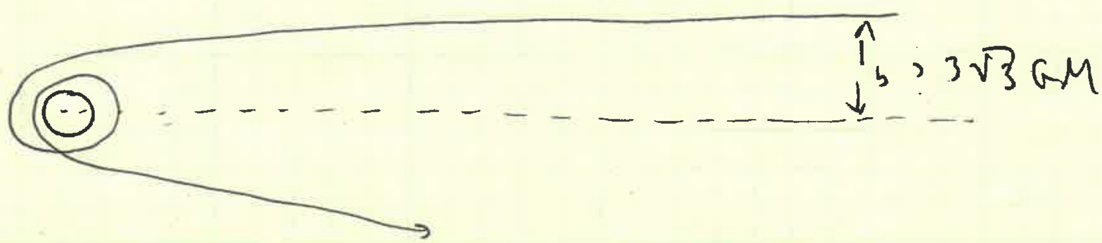
Results:

$b > 3\sqrt{3}\, GM$. $\left(\frac{dr}{d\lambda}\right)^2 = 0$ at some value of r.

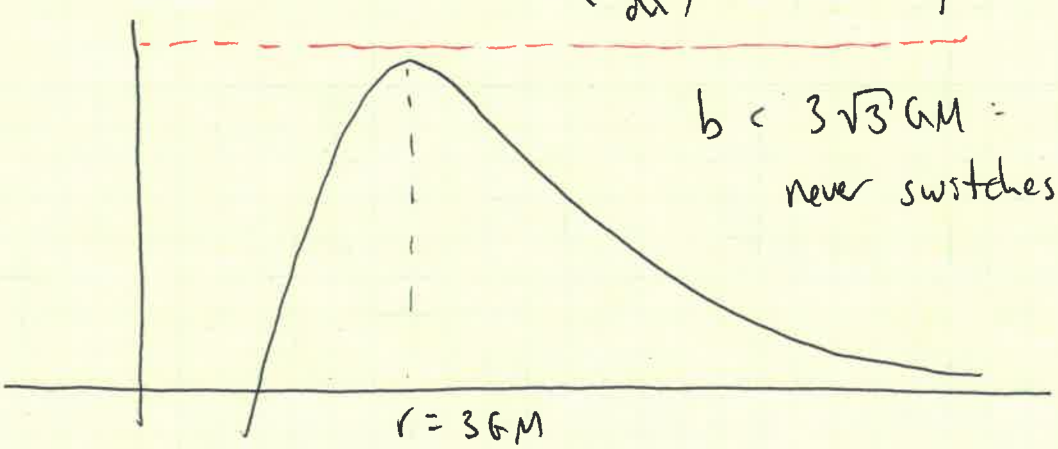
Light ray comes in, bends, goes back out.



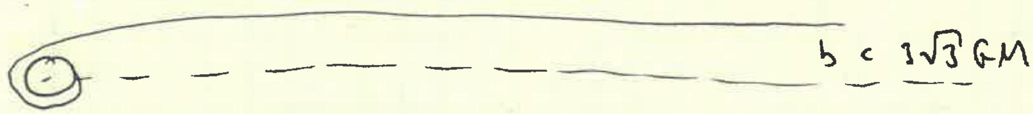
$b > 3\sqrt{3}\, GM$: $dr/d\lambda$ switches sign at some $r > 3GM$



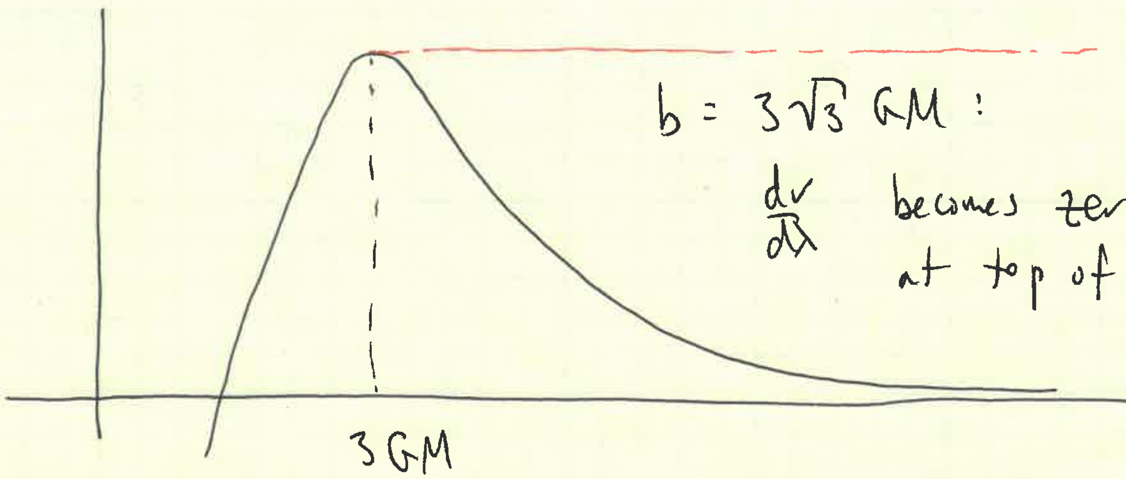
$b < 3\sqrt{3}\, GM$: $\left(\frac{dr}{d\lambda}\right)^2 > 0$ for all r.



$b < 3\sqrt{3}\, GM$: $dr/d\lambda$ never switches sign.



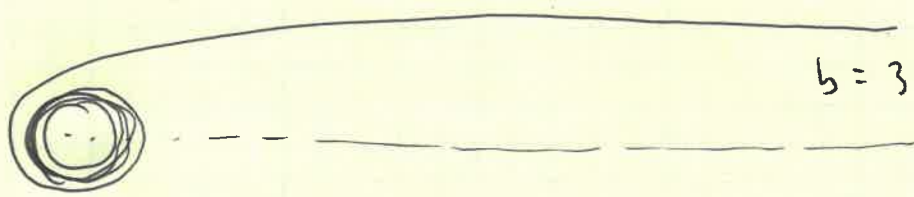
$b = 3\sqrt{3}\,GM$: $\left(\frac{dr}{d\lambda}\right) = 0$ at $r = 3GM$



$b = 3\sqrt{3}\,GM$:

$\frac{dr}{d\lambda}$ becomes zero exactly at top of potential.

In this case, the light ray is pulled into an unstable circular orbit exactly at $r = 3GM$:



$b = 3\sqrt{3}\,GM$

Generalizing to Kerr ... crucial since the expectation is that this is the generic solution describing gravitational collapse's outcome in our universe.

$$ds^2 = -\left(1 - \frac{2GMr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2$$

$$+ \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2$$

$$- \frac{2GMar\sin^2\theta}{\rho^2}\left(dt\,d\phi + d\phi\,dt\right)$$

$$\rho = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2GMr + a^2$$

$$a = |\vec{S}|/M$$

Since $\partial_t g_{\mu\nu} = 0 = \partial_\phi g_{\mu\nu}$, we have two conserved constants:

$$E \equiv -p_t = -m\left(g_{tt}\frac{dt}{d\tau} + g_{t\phi}\frac{d\phi}{d\tau}\right)$$

$$L_z \equiv p_\phi = m\left(g_{\phi\phi}\frac{d\phi}{d\tau} + g_{\phi t}\frac{dt}{d\tau}\right)$$

Less obviously, there exists a Killing tensor $Q_{\mu\nu}$, which allows us to deduce the existence of a 3RD conserved constant:

$$Q \equiv Q_{\mu\nu}p^\mu p^\nu$$

$$= p_\theta^2 + \cos^2\theta\left[a^2(m^2 - E^2) + \csc^2\theta L_z^2\right]$$

When $a \to 0$, $Q = L_x^2 + L_y^2$ - the "rest" of the total angular momentum. Not quite as clean an association for Kerr due to lack of spherical symmetry.

Because a geodesic has 4-conserved quantities - (E, L_z, Q, and $m = \sqrt{-p^\mu p_\mu}$) - it turns out to be possible to separate the equations of motion

(Go to slides)

Set $m = 0$, change ~~[illegible]~~ $m\, d/d\tau \to d/d\lambda$, equations describe light propagating in the spacetime.

Methods for analyzing dynamical spacetimes with less or no symmetry or small parameter.

The techniques we have studied in depth have relied upon either approximations to the full equations, or we have assumed special symmetry.

The results of this are surprisingly robust! The weak-field limit encompasses many important physical phenomena (gravitational lensing, solar system, galactic environment, many binary stars); symmetry lets us describe the large-scale structure of the universe, black holes, neutron stars.

We nonetheless still must do better for some important problems. ~~Two~~ Three approaches:

① Iterating from weak to not-so-weak fields.

② Examining perturbations about exact solutions.
→ I will focus on black holes as exact solution; similar ideas work for cosmological spacetimes and I'll post references.

③ Direct numerical integration of Einstein field equations.

Iteration from weak-field: Post-Newtonian theory
(Lecture based on "Gravity" by Poisson + Will, and Luc Blanchet, Living Reviews in Relativity, v17, p 2, 2014.)

Einstein field equation normally written

$$G^{\alpha\beta}[g, \partial g, \partial^2 g] = 8\pi G T^{\alpha\beta}$$

Post-Newtonian begins by changing to a variable

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$$

a tensor density. This allows us to write the field equations EXACTLY as

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = 16\pi G\left[(-g)T^{\alpha\beta} + \frac{\Lambda^{\alpha\beta}}{16\pi G}\right]$$

(Note: $\Lambda^{\alpha\beta} \equiv 16\pi G(-g)\, t^{\alpha\beta}_{L-L}$, where $t^{\alpha\beta}_{L-L}$ is the Landau-Lifschitz pseudotensor, is another common form of notation.)

Here, $H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}$

Amazingly, this form only involves partial derivatives ... simpler in some ways.

To proceed, we define

$$h^{\alpha\beta} \equiv \mathfrak{g}^{\alpha\beta} - \eta^{\alpha\beta}$$

This means $g^{\alpha\beta} = (-g)^{-1/2}\left(\eta^{\alpha\beta} + h^{\alpha\beta}\right)$

We are not necessarily assuming $\|h^{\alpha\beta}\| \ll 1$!

We impose a gauge condition : $\partial_\alpha h^{\alpha\beta} = 0$

"Harmonic coordinates" or "deDonder gauge"

In this gauge, the field equations become

$$\Box h^{\alpha\beta} = 16\pi G \tau^{\alpha\beta}$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ → flat spacetime wave operator!

and $\tau^{\alpha\beta} = (-g)T^{\alpha\beta} + \Lambda^{\alpha\beta}/16\pi G$

The formal solution is trivial:

$$h^{\alpha\beta}(\underset{\sim}{x}, t) = -4G \int \frac{\tau^{\alpha\beta}(\underset{\sim}{x}', t - |\underset{\sim}{x} - \underset{\sim}{x}'|)}{|\underset{\sim}{x} - \underset{\sim}{x}'|} d^3x'$$

However, $\Lambda^{\alpha\beta}$ depends on solution $h^{\alpha\beta}$!

$$\Lambda^{\alpha\beta} = N^{\alpha\beta}(h,h) + M^{\alpha\beta}(h,h,h) + L^{\alpha\beta}(h,h,h,h) + \cdots$$

An integro-differential equation for $h^{\alpha\beta}$.

Path to success: use G as a "small parameter" to gather terms and organize the solution.

Allows us to solve the equations <u>iteratively</u>:

$$h^{\alpha\beta} = \sum_{n=1}^{\infty} G^n h_n^{\alpha\beta}$$

Field equations hold at each order in n ... terms at order $m < n$ act as sources at order n.