8.962 LECTURE 24

Wrap up of black hole orbits:
Motion of light; orbits in Kerr

Going beyond symmetry.

Image of radiation emitted by material flowing onto the black hole at the core of galaxy M87. Image produced using large-scale radio interferometry project known as the Event Horizon Telescope

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Recap: Examined motion of a massive body in Schwarzschild spacetime … It is governed by the equations

$$\frac{dt}{d\tau} = \frac{\hat{E}}{1 - 2GM/r}$$

$$\frac{d\phi}{d\tau} = \hat{L}/r^2$$

$$\left(\frac{dr}{d\tau}\right)^2 = \hat{E}^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right) \equiv \hat{E}^2 - V_{\rm eff}(r)$$

Pick energy and angular momentum; characteristics of effective potential and energy determine character of the orbit.

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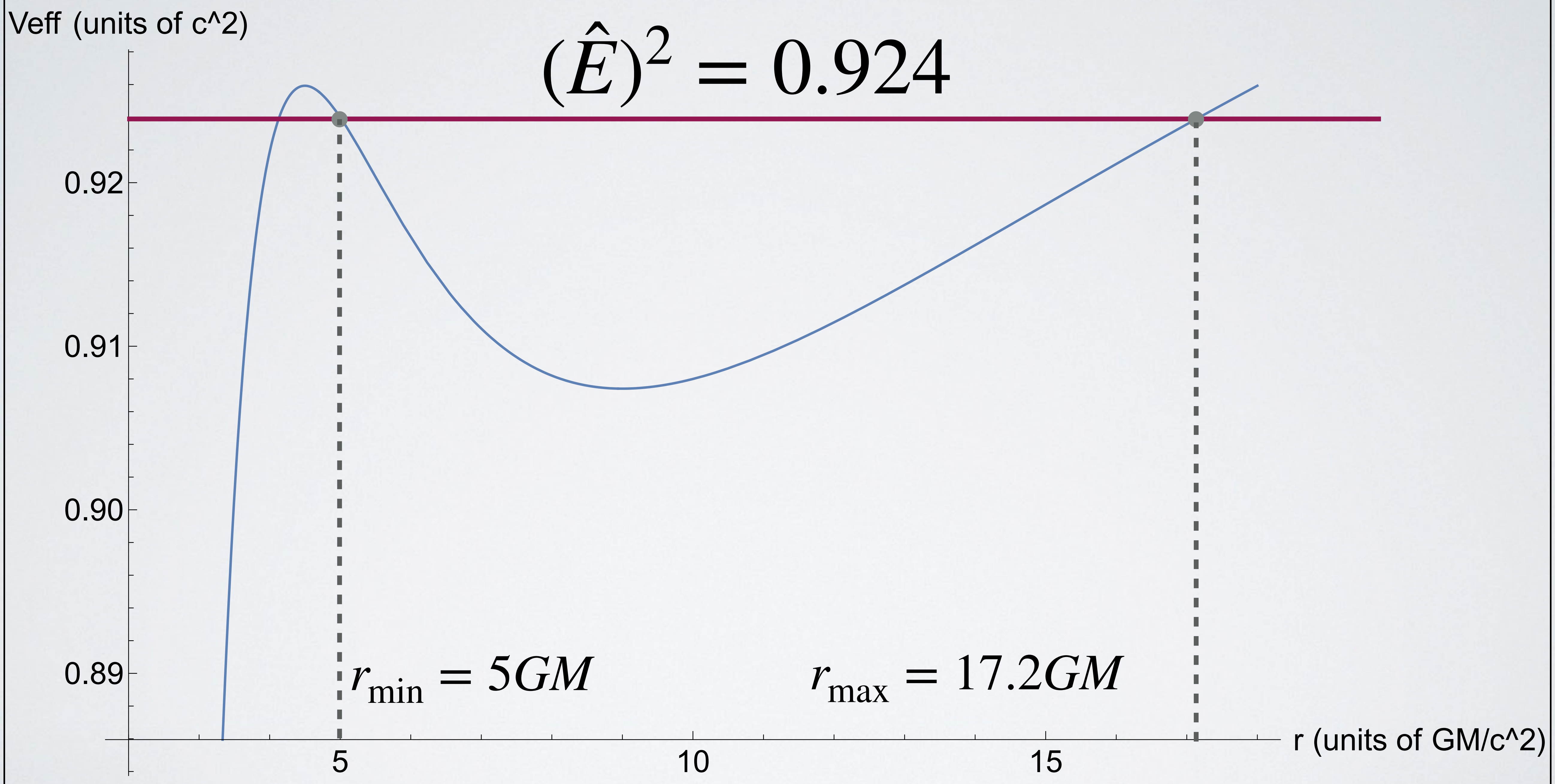

Example: Eccentric orbit.

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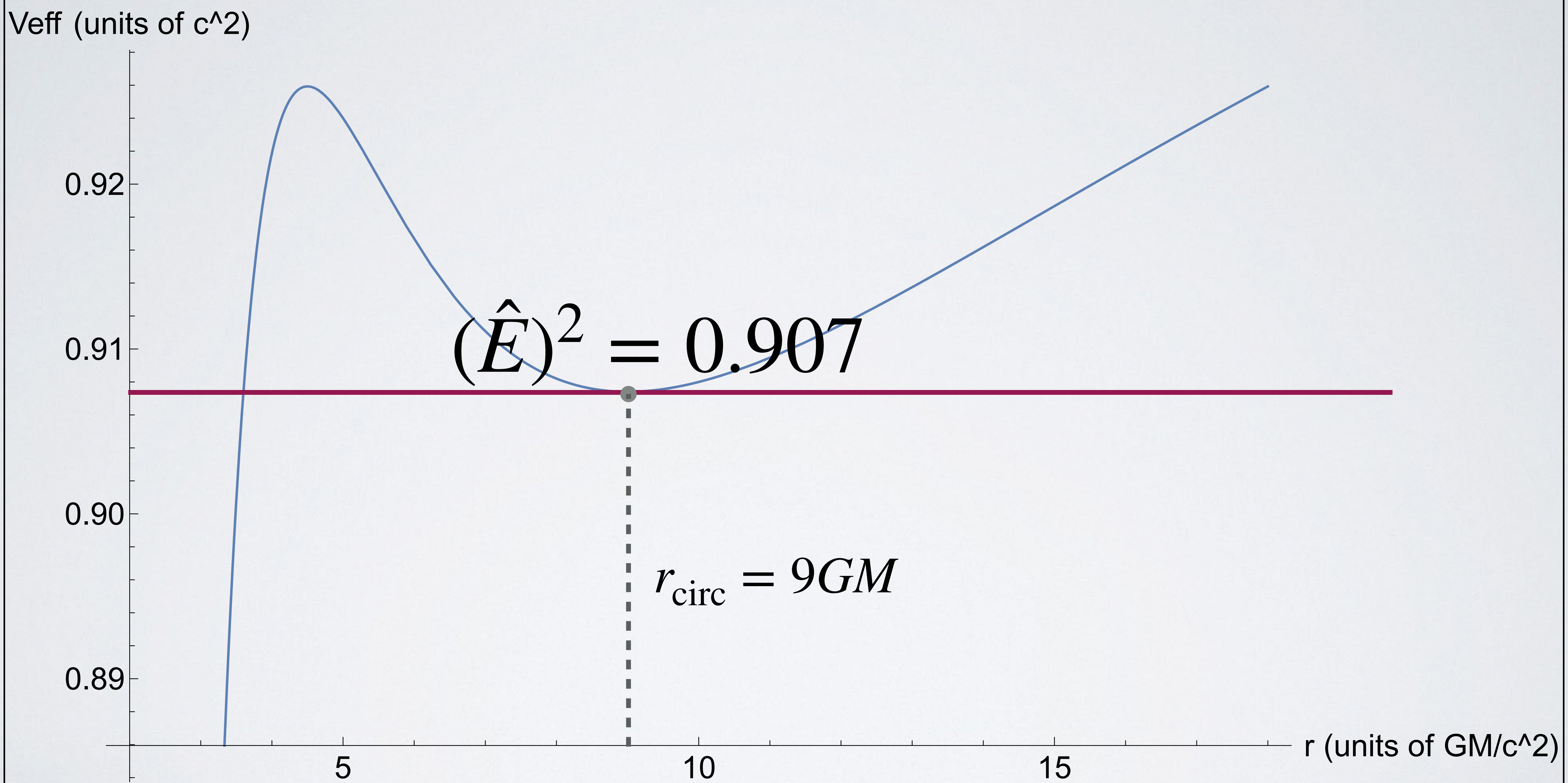

Example: Circular orbit.

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Circular condition is that the orbit have $dr/d\tau = 0$ and sit at the minimum of $V_{\rm eff}(r)$; imposing these conditions leads to the solutions

$$\hat{E} = \frac{1 - 2GM/r}{\sqrt{1 - 3GM/r}}$$

$$\hat{L} = \pm\sqrt{\frac{GMr}{1 - 3GM/r}}$$

Require in addition that the minimum be stable:

$$\frac{\partial^2 V_{\rm eff}}{\partial r^2} > 0 \quad \longrightarrow \quad r > 6GM\,.$$

Leads to a very non-Newtonian characteristic: an "innermost stable orbit." No stable circular orbits exist inside the radius $r = 6GM$.

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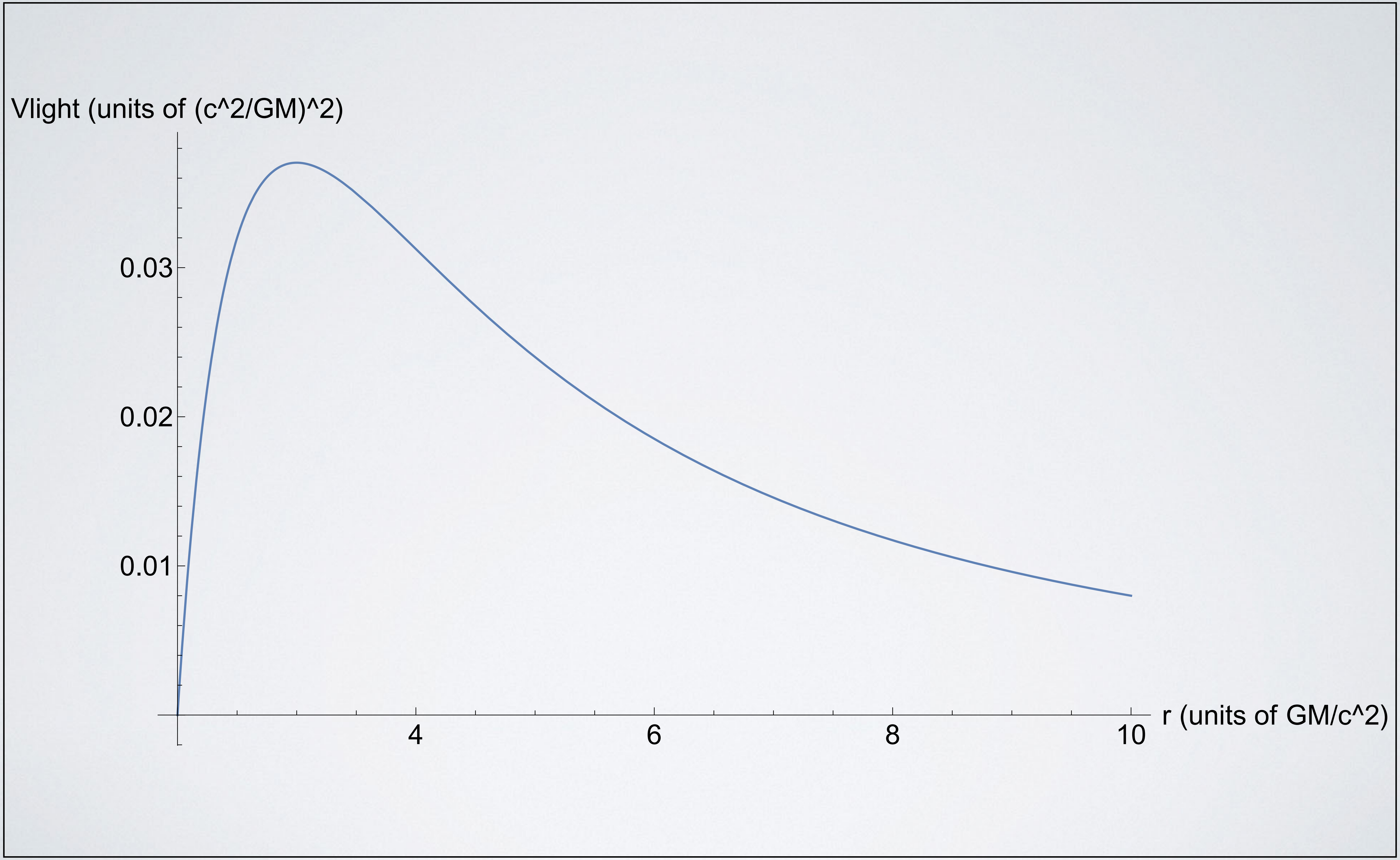

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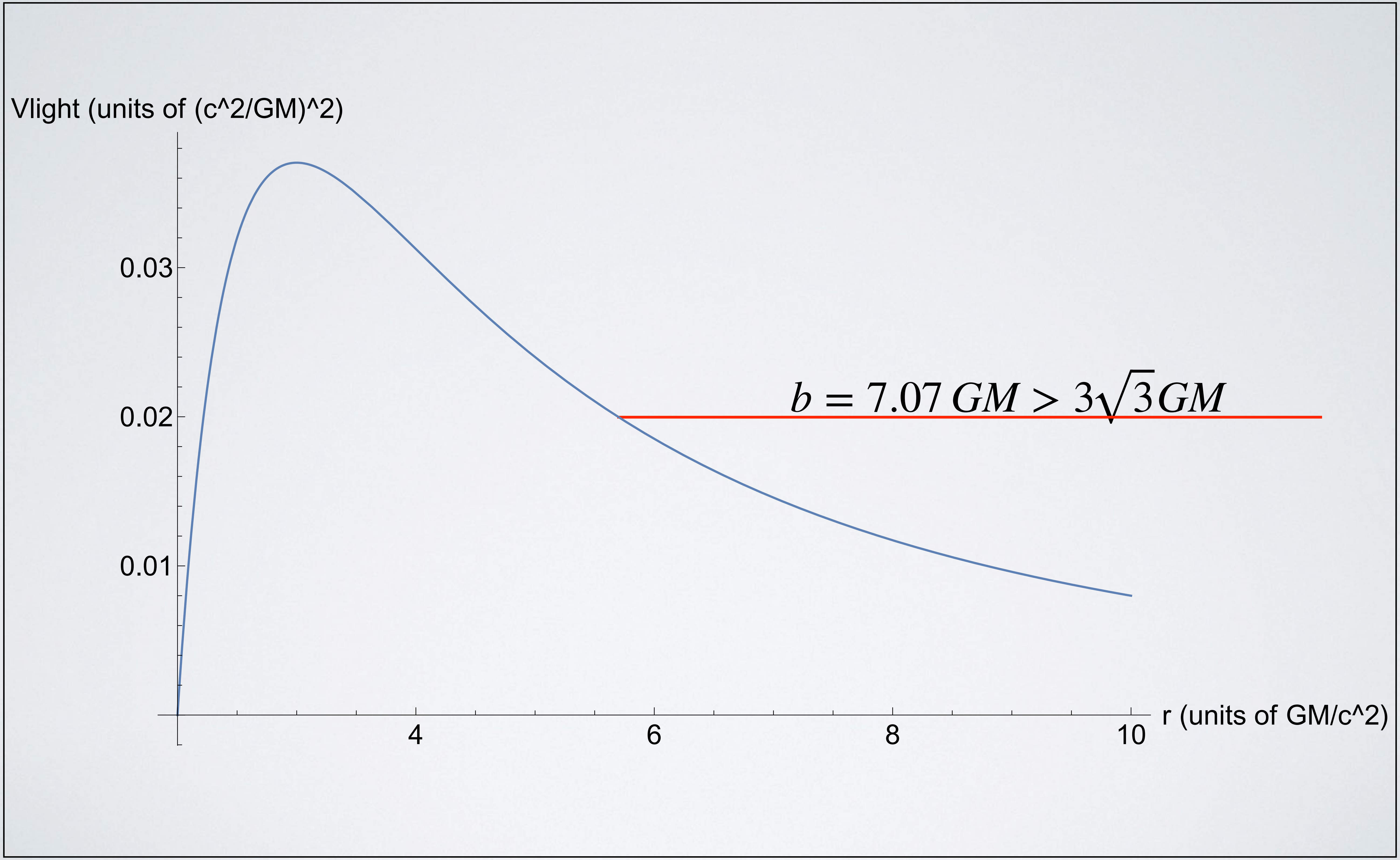

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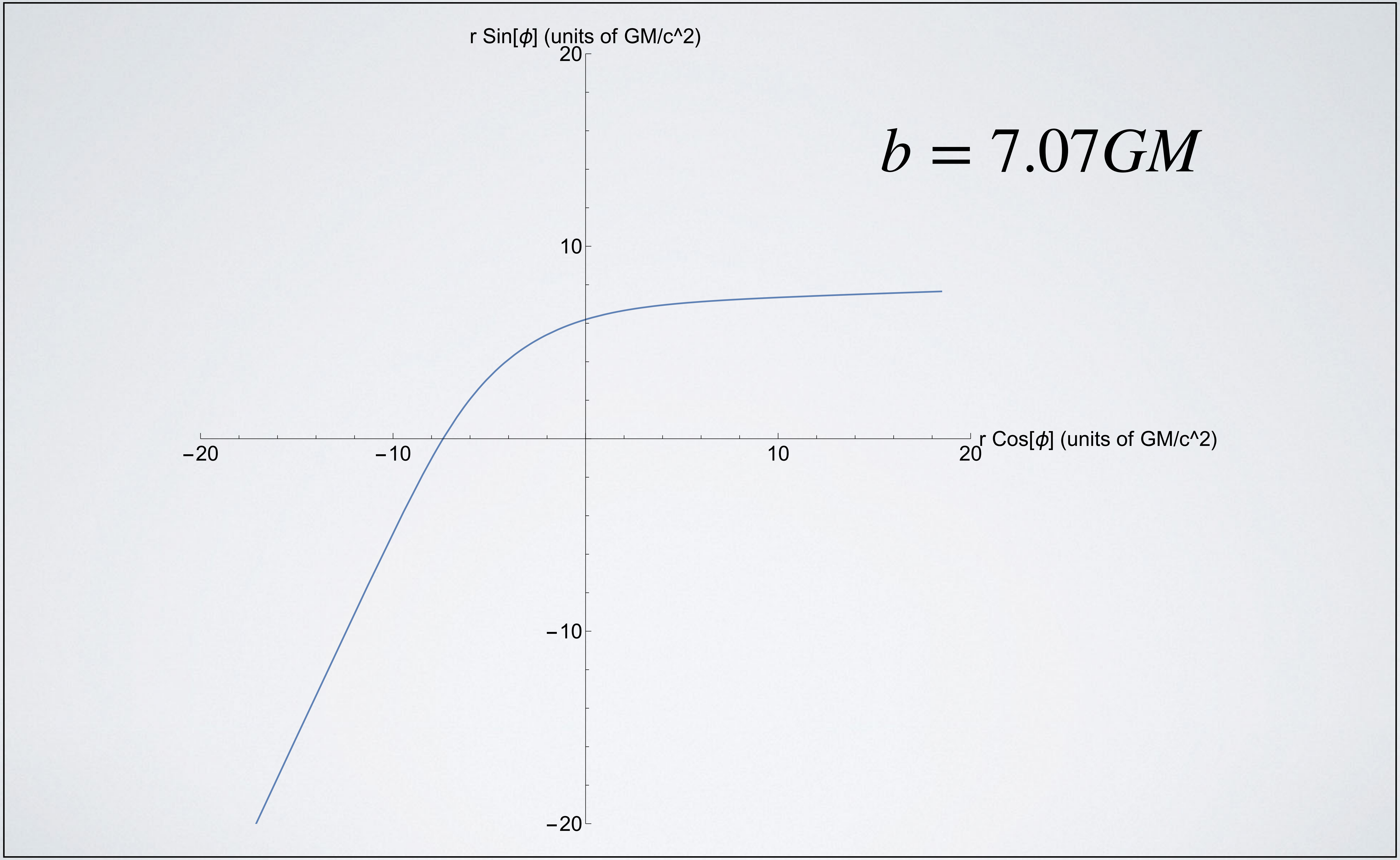

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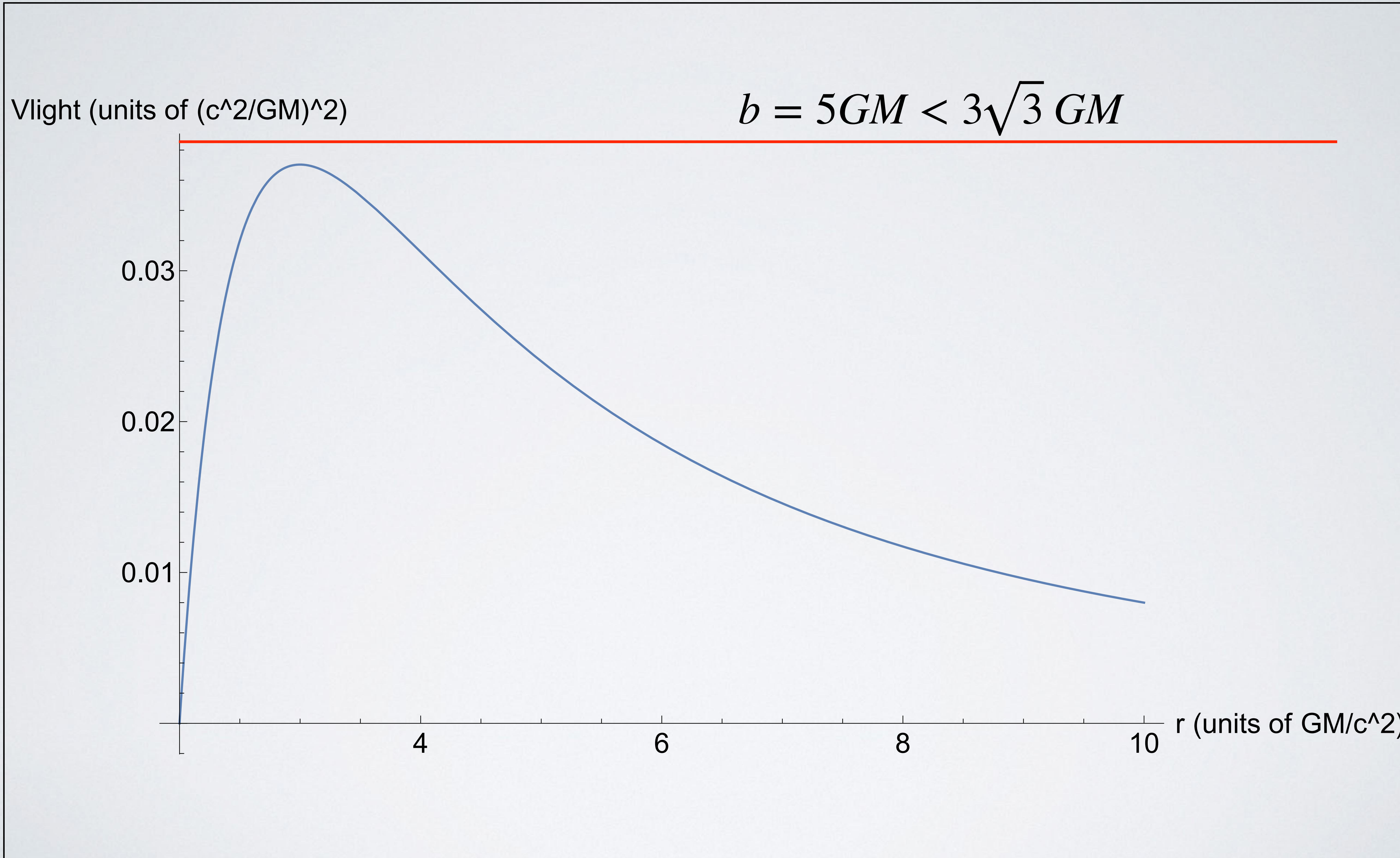

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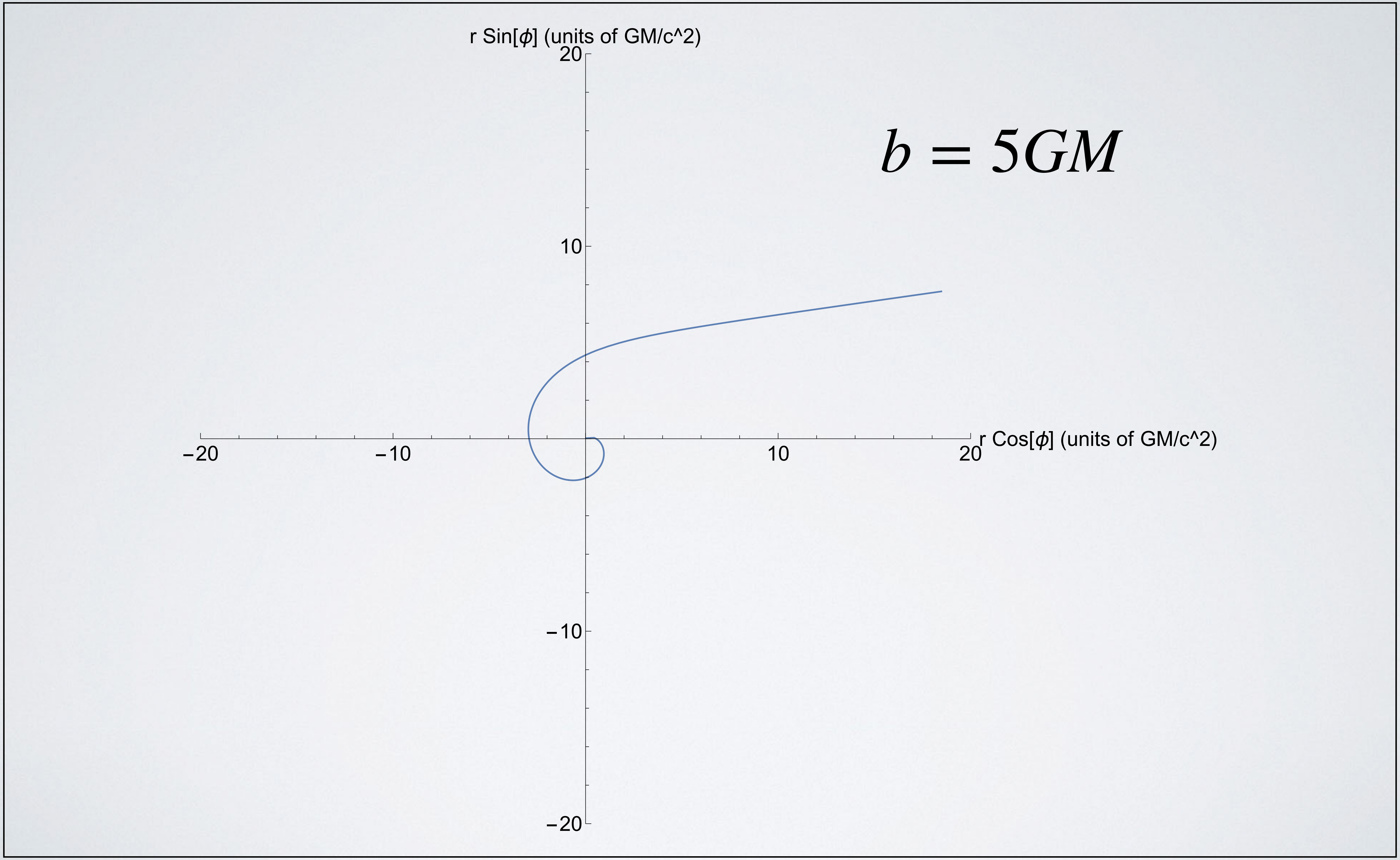

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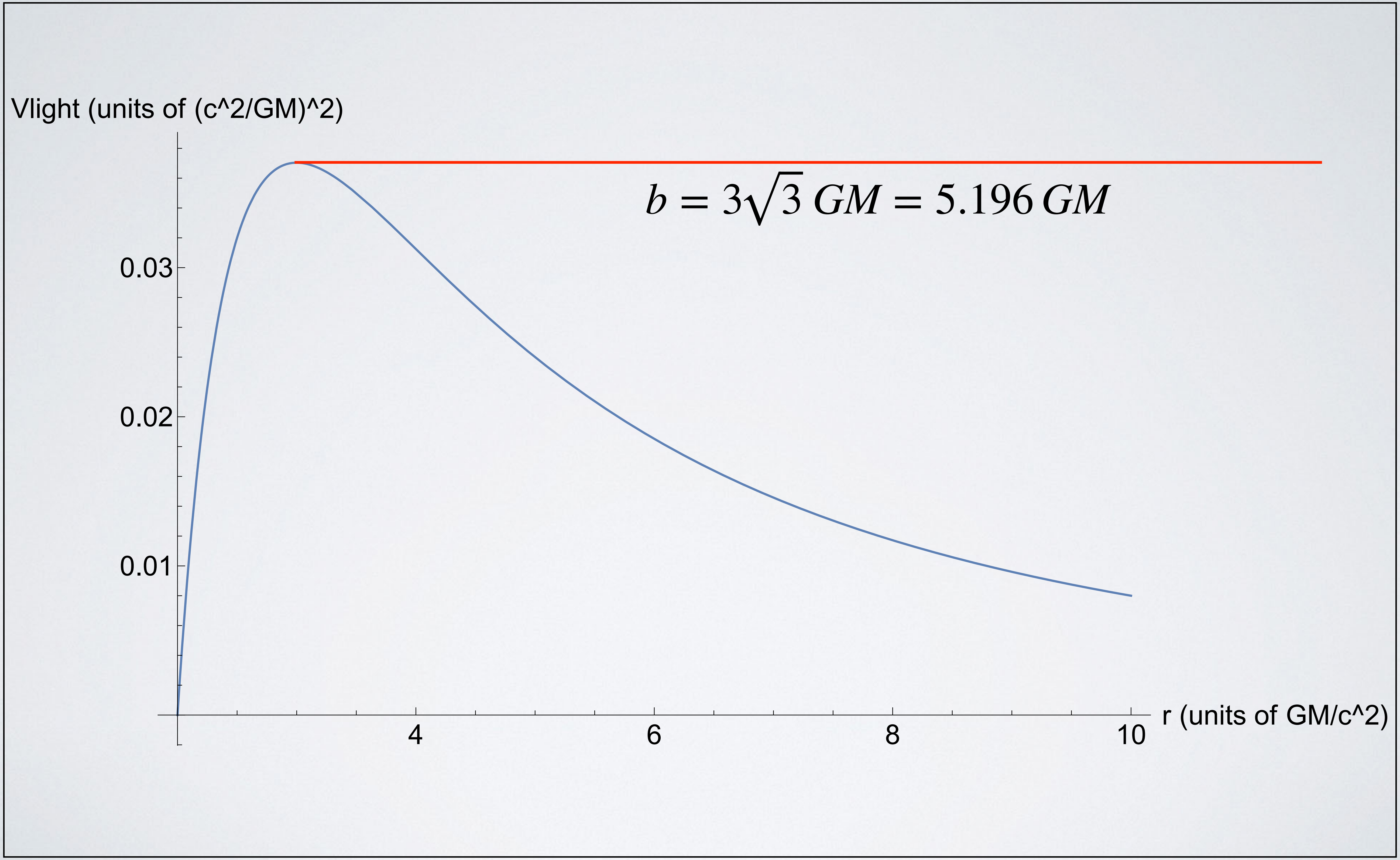

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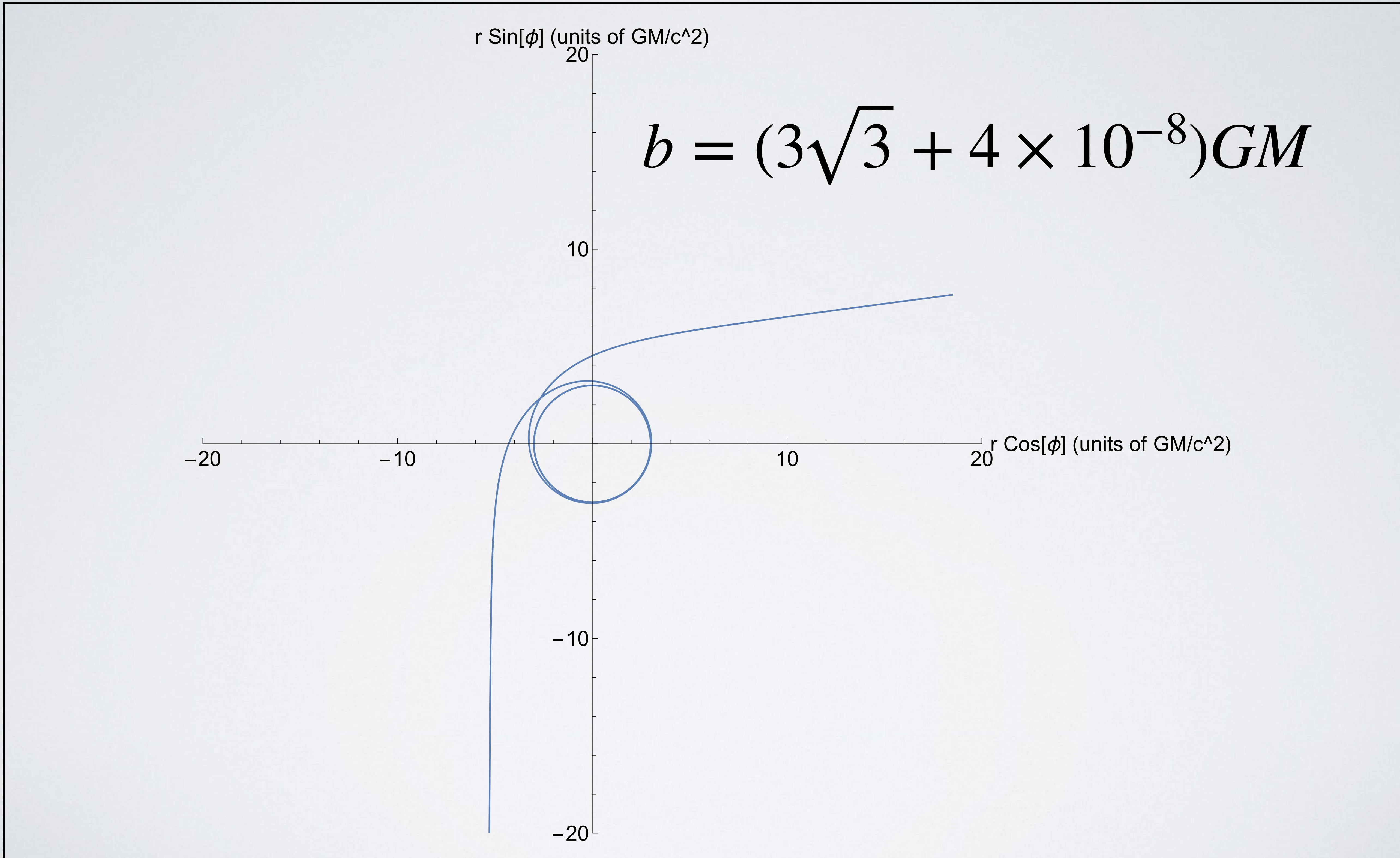

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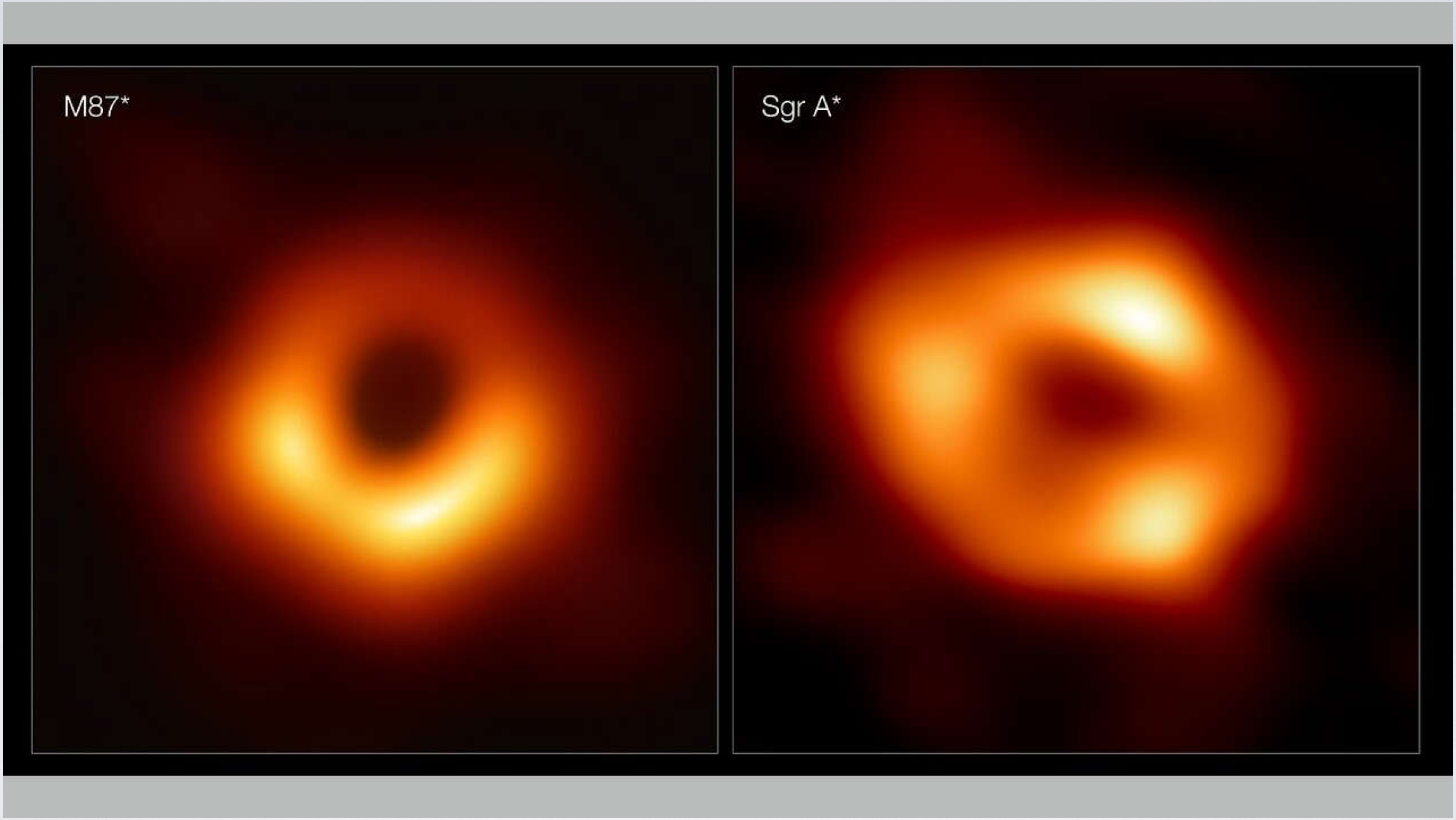

Radiation emitted by material accreting onto two famous BHs

8.962 LECTURE 24

Kerr motion: Conserved constants make it possible to separate the equations of motion. Using $\Sigma = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - 2GMr + a^2$, plus the definition $a = |\mathbf{S}|/GM$, we have

$$\Sigma^2\left(\frac{dr}{d\tau}\right)^2 = \left(\hat{E}(r^2+a^2) - a\hat{L}_z\right)^2 - \Delta\left(r^2 + (\hat{L}_z - a\hat{E})^2 + \hat{Q}\right) \equiv R(r)$$

$$\Sigma^2\left(\frac{d\theta}{d\tau}\right)^2 = \hat{Q} - \hat{L}_z^2\cot^2\theta - a^2\cos^2\theta(1-\hat{E}^2) \equiv \Theta(\theta)$$

$$\Sigma\left(\frac{d\phi}{d\tau}\right) = \csc^2\theta\hat{L}_z + \frac{a}{\Delta}(2Mr\hat{E} - a\hat{L}_z)$$

$$\Sigma\left(\frac{dt}{d\tau}\right) = \hat{E}\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right] - \frac{2Mra}{\Delta}\hat{L}_z$$

Pick energy, axial angular momentum, "Carter constant" Q: determines the properties of bound orbits.

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Useful subset: $Q = 0$ requires $\theta = \pi/2$, so these are *equatorial* orbits. If we further set $R = 0$, $dR/dr = 0$, we have *circular* equatorial orbits.

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Stable circular equatorial orbits further have $d^2R/dr^2 > 0$; solve for the system ($R = 0$, $dR/dr = 0$, $d^2R/dr^2 = 0$), generalize the innermost stable circular orbit to Kerr. Result:

$$r_{\rm ISCO}/GM = 3 + Z_2 \mp \left[(3 - Z_1)(3 + Z_1 + 2Z_2)\right]^{1/2}$$

$$Z_1 = 1 + \left(1 - q^2\right)^{1/3}\left[(1+q)^{1/3} + (1-q)^{1/3}\right]$$

$$Z_2 = \left(3q^2 + Z_1^2\right)^{1/2} \qquad (q \equiv a/GM)$$

Do this exercise for a light-like trajectory:

$$r_{\rm phot}/GM = 2 + 2\cos\left[\frac{2}{3}\cos^{-1}(\mp q)\right]$$

Reference: Bardeen, Press, Teukolsky, Astrophys. J. **178**, 347 (1972).

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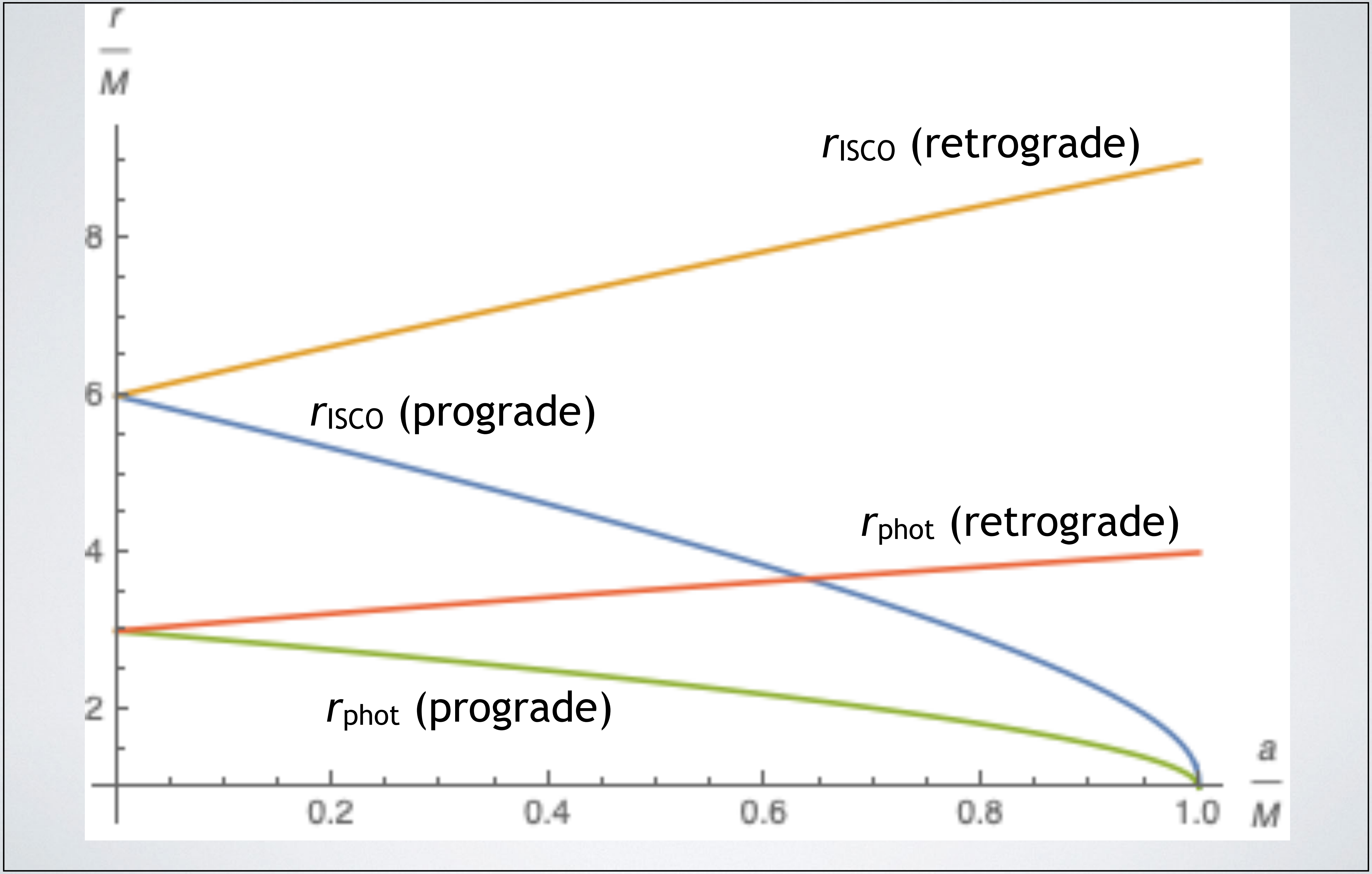

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Quantity which appears in the "iterative" form of the field equations that we use to define post-Newtonian theory:

$$\begin{aligned}\Lambda^{\alpha\beta} &= -h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda}\\ &\quad - g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu}\\ &\quad + \frac{1}{8}\left(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu}\right)\left(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi}\right)\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}\,.\end{aligned}$$