Methods for analyzing dynamical spacetimes with less or no symmetry or small parameter.

The techniques we have studied in depth have relied upon either approximations to the full equations, or we have assumed special symmetry.

The results of this are surprisingly robust! The weak-field limit encompasses many important physical phenomena (gravitational lensing, solar system, galactic environment, many binary stars); symmetry lets us describe the large-scale structure of the universe, black holes, neutron stars.

We nonetheless still must do better for some important problems. ~~Two~~ Three approaches:

1. Iterating from weak to not-so-weak fields.
2. Examining perturbations about exact solutions.
   → I will focus on black holes as exact solution; similar ideas work for cosmological spacetimes, and I'll post references.
3. Direct numerical integration of Einstein field equations.

Iteration from weak-field : Post-Newtonian theory

(Lecture based on "Gravity" by Poisson + Will, and Luc Blanchet, Living Reviews in Relativity, v17, p 2, 2014.)

Einstein field equation normally written

$$G^{\alpha\beta}[g, \partial g, \partial^2 g] = 8\pi G T^{\alpha\beta}$$

Post-Newtonian begins by changing to a variable

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$$

a tensor <u>density</u>. This allows us to write the field equations EXACTLY as

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = 16\pi G\left[(-g)T^{\alpha\beta} + \frac{\Lambda^{\alpha\beta}}{16\pi G}\right]$$

(Note: $\Lambda^{\alpha\beta} \equiv 16\pi G(-g)\, t^{\alpha\beta}_{L\text{-}L}$, where $t^{\alpha\beta}_{L\text{-}L}$ is the Landau-Lifschitz pseudotensor, is another common form of notation.)

Here, $H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu}$

Amazingly, this form only involves partial derivatives ... simpler in some ways.

To proceed, we define

$$h^{\alpha\beta} \equiv g^{\alpha\beta} - \eta^{\alpha\beta}$$

This means $\quad g^{\alpha\beta} = (-g)^{-1/2}\left(\eta^{\alpha\beta} + h^{\alpha\beta}\right)$

We are not necessarily assuming $\|h^{\alpha\beta}\| \ll 1$!

We impose a gauge condition : $\partial_\alpha h^{\alpha\beta} = 0$

"Harmonic coordinates" or "de Donder gauge"

In this gauge, the field equations become

$$\Box h^{\alpha\beta} = 16\pi G\, \tau^{\alpha\beta}$$

where $\quad \Box = \eta^{\mu\nu}\partial_\mu\partial_\nu \rightarrow$ flat spacetime wave operator!

and $\quad \tau^{\alpha\beta} = (-g)\, T^{\alpha\beta} + \Lambda^{\alpha\beta}/16\pi G$

The formal solution is trivial:

$$h^{\alpha\beta}(\underset{\sim}{x}, t) = -4G \int \frac{\tau^{\alpha\beta}(\underset{\sim}{x}', t - |\underset{\sim}{x} - \underset{\sim}{x}'|)}{|\underset{\sim}{x} - \underset{\sim}{x}'|}\, d^3x'$$

However, $\Lambda^{\alpha\beta}$ depends on solution $h^{\alpha\beta}$!

$$\Lambda^{\alpha\beta} = N^{\alpha\beta}(h,h) + M^{\alpha\beta}(h,h,h) + L^{\alpha\beta}(h,h,h,h)$$

$$+ \cdots$$

An integro-differential equation for $h^{\alpha\beta}$.

Path to success: use G as a "small parameter" to gather terms and organize the solution.

Allows us to solve the equations <u>iteratively</u>:

$$h^{\alpha\beta} = \sum_{n=1}^{\infty} G^n h_n^{\alpha\beta}$$

Field equations hold at each order in n ... terms at order $m < n$ act as sources at order n.

Perturbation theory: conceptually similar to weak field analysis.

$$g_{\alpha\beta} = g^{B}_{\alpha\beta} + h_{\alpha\beta}, \quad \|h_{\alpha\beta}\| / \|g^{B}_{\alpha\beta}\| \ll 1.$$

Expand Einstein to linear order in h, develop a "wave equation" for the perturbation.

Consider <u>vacuum</u> perturbations of Schwarzschild:

$$^{(B)}ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

Vacuum means $G_{\alpha\beta} = 0 \rightarrow R_{\alpha\beta} = 0$.

$$\rightarrow \hat{R}_{\alpha\beta} + \delta R_{\alpha\beta} = 0$$

$\hookrightarrow$ automatically satisfied since background is a vacuum solution.

$\rightarrow$ $\delta R_{\alpha\beta} = 0$ is our wave equation.

$$\rightarrow \hat{\nabla}_\beta \left(\delta \Gamma^{\gamma}_{\alpha\gamma}\right) - \hat{\nabla}_\gamma \left(\delta \Gamma^{\gamma}_{\alpha\beta}\right) = 0$$

$$\delta \Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}_{B} \left(\hat{\nabla}_\beta h_{\alpha\delta} + \hat{\nabla}_\alpha h_{\beta\delta} - \hat{\nabla}_\delta h_{\alpha\beta}\right)$$

Before grinding, pause & examine symmetries. Since Schw. is spherically symmetric, should find a harmonic expansion to be useful:

$$h_{\alpha\beta} = \begin{cases} h_{00} & - \text{ scalar harmonics} \\ h_{0i} & - \text{ vector harmonics} \\ h_{ij} & - \text{ tensor harmonics.} \end{cases}$$

Vector & tensor harmonics will come in two different parity varieties:

Example: $$h_{ij} = \sum_{lm} \left[a_{lm} \underset{\uparrow \text{ even}}{A^{lm}_{ij}}(\theta,\phi) + b_{lm} \underset{\uparrow \text{ odd}}{B^{lm}_{ij}}(\theta,\phi) \right]$$

$$P(\mathcal{Y}(\theta,\phi)) = \mathcal{Y}(\pi-\theta, \phi+\pi)$$
$$= (-1)^l \mathcal{Y}(\theta,\phi) \quad \text{"even", "polar"}$$
$$= (-1)^{l+1} \mathcal{Y}(\theta,\phi) \quad \text{"odd", "axial"}.$$

Focus on odd parity. Each mode (l,m) contributes as

$$h_{00} = 0 \qquad \text{(spherical harmonics are polar!)}$$

$$h_{0i} = H_0(t,r)\left[0, -\csc\theta\, \partial_\phi Y_{lm}, \sin\theta\, \partial_\theta Y_{lm}\right] \quad \text{(even)}$$

$$h_{ij} = H_1(t,r)\left[\hat{e}_1\right]_{ij} + H_2(t,r)\left[\hat{e}_2\right]_{ij}$$

$$\left[\hat{e}_1\right]_{ij} = \begin{bmatrix} 0 & -\csc\theta\, \partial_\phi Y_{lm} & \sin\theta\, \partial_\theta Y_{lm} \\ -\csc\theta\, \partial_\phi Y_{lm} & 0 & 0 \\ \sin\theta\, \partial_\theta Y_{lm} & 0 & 0 \end{bmatrix}$$

$\left[\hat{e}_2\right]_{ij} =$ mess involving 2nd derivs of Y_{lm}.

2 Further simplifications:

1. Focus on $m=0 \rightarrow \partial_\phi = 0$. Not necessary - axisymmetry means get same wave equation for all m. But cleans up.

2. Set $H_2(t,r) = 0 \rightarrow$ "Regge-Wheeler gauge".

Result: $$h_{\alpha\beta} = \begin{bmatrix} 0 & 0 & 0 & H_0 \\ 0 & 0 & 0 & H_1 \\ 0 & 0 & 0 & 0 \\ H_0 & H_1 & 0 & 0 \end{bmatrix} \sin\theta\, \partial_\theta P_\ell$$

Legendre polynomial

Run this through $\delta R_{\alpha\beta} = 0$. Result:

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} + \left(1 - \frac{2GM}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6GM}{r^3}\right] Q = 0$$

$$Q = \frac{H_1}{r}\left(1 - \frac{2GM}{r}\right)$$

$$\frac{\partial H_0}{\partial t} = \frac{\partial}{\partial r_*}\left(r_* Q\right)$$

$$r_* = r + 2GM \ln\left[\frac{r}{2GM} - 1\right]$$

"Regge-Wheeler" equation. Same exercise for even parity modes - Zerilli equation. (Similar, but messier potential.)

Note asymptotic solutions: $\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} = 0$

for $r \to \infty$, $r \to 2GM$: $Q \sim \exp\left[i\omega(t \pm r_*)\right]$

Add physical boundary conditions:

$$Q \sim \exp\left[i\omega(t - r_*)\right] \quad r \to \infty \quad \to \text{OUTGOING}$$

$$Q \sim \exp\left[i\omega(t + r_*)\right] \quad r \to 2GM \quad \to \text{INGOING.}$$

Kerr? Unfortunately, a perturbative metric expansion doesn't work well: Lack of spherical symmetry means we don't get nicely separated equations. Resulting PDEs not much better than "raw" Einstein field equations.

By a miracle, "nice" equations result by focusing on <u>curvature</u>. Key ingredients:

- Bianchi identity, transformed into a wave equation:

$$\nabla^\gamma \left[\nabla_\gamma R_{\alpha\beta\mu\nu} + \nabla_\alpha R_{\beta\gamma\mu\nu} + \nabla_\beta R_{\gamma\alpha\mu\nu} \right] = 0$$

- Vacuum solution, so Riemann goes over to Weyl tensor $C_{\alpha\beta\gamma\delta}$.

- Introduce a tetrad of null vectors adapted to radiation:

$$l_\alpha \doteq \left[1, -\frac{\Sigma}{\Delta}, 0, -a\sin^2\theta \right]$$ "outgoing" ~~"ingoing"~~

$$n_\alpha \doteq \frac{1}{2}\left[\frac{\Delta}{\Sigma}, 1, 0, -\frac{a\Delta\sin^2\theta}{\Sigma} \right]$$ ~~"outgoing"~~ "ingoing"

$$m_\alpha \doteq \frac{\rho^{-1}}{\sqrt{2}}\left[ia\sin\theta, 0, \Sigma, -i(r^2+a^2)\sin\theta \right]$$

"angular"

$\bar{m}_\alpha$ = complex conjugate of m_α

$$\rho = -\left(\frac{1}{r - ia\cos\theta} \right)$$

Using this tetrad, map the 10 components of the Weyl tensor to 5 complex scalars:

$$\Psi_0 = -C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta l^\gamma m^\delta$$
$$\Psi_1 = -C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta l^\gamma m^\delta$$
$$\Psi_2 = -C_{\alpha\beta\gamma\delta}\, l^\alpha m^\beta \bar{m}^\gamma n^\delta$$
$$\Psi_3 = -C_{\alpha\beta\gamma\delta}\, l^\alpha n^\beta \bar{m}^\gamma n^\delta$$
$$\Psi_4 = -C_{\alpha\beta\gamma\delta}\, n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$$

(Framework developed by Ezra Newman and Roger Penrose.)

For the Kerr metric, $\Psi_0 = \Psi_1 = \Psi_3 = \Psi_4 = 0$

$$\Psi_2 = \frac{-GM}{(r - ia\cos\theta)^3}$$

If you perturb Kerr, find that Ψ_0 describes radiation that is ingoing (toward the event horizon), Ψ_4 describes radiation that is outgoing (toward infinity).

Equations governing different Ψ_x allow us to separate out how different contributions to the curvature depend on the perturbation.

Particularly fruitful: Analysis of Ψ_0 + Ψ_4 allows us to fully characterize radiation in spacetime (Teukolsky 1973).

Generate the "curvature wave equation" for Φ_4; turns out to separate upon introducing the multipolar expansion

$$\Phi_4 = \frac{1}{(r - ia\cos\theta)^4} \int d\omega \sum_{lm} R_{lm\omega}(r)\, S_{lm\omega}(\theta) \times e^{im\phi} e^{-i\omega t}$$

$S_{lm\omega}(\theta)$ are a family of functions called "spin-weighted spheroidal harmonics" - conceptually similar to spherical harmonics.

$R_{lm\omega}(r)$ solves an ordinary differential equation:

$$\Delta^2 \frac{d}{dr}\left[\frac{1}{\Delta}\frac{dR}{dr}\right] - V_{lm\omega} R = T_{lm\omega}(r)$$

$$V_{lm\omega}(r) = 8i\omega r + \lambda - \frac{(K^2 + 4i(r - GM)K)}{\Delta}$$

$$K = (r^2 + a^2)\omega - ma$$

$\lambda =$ eigenvalue of spheroidal harmonic
$= (l+2)(l-1)$ when $a = 0$

$T_{lm\omega} \rightarrow$ built from $T^{\mu\nu}$ of whatever is perturbing our BH.