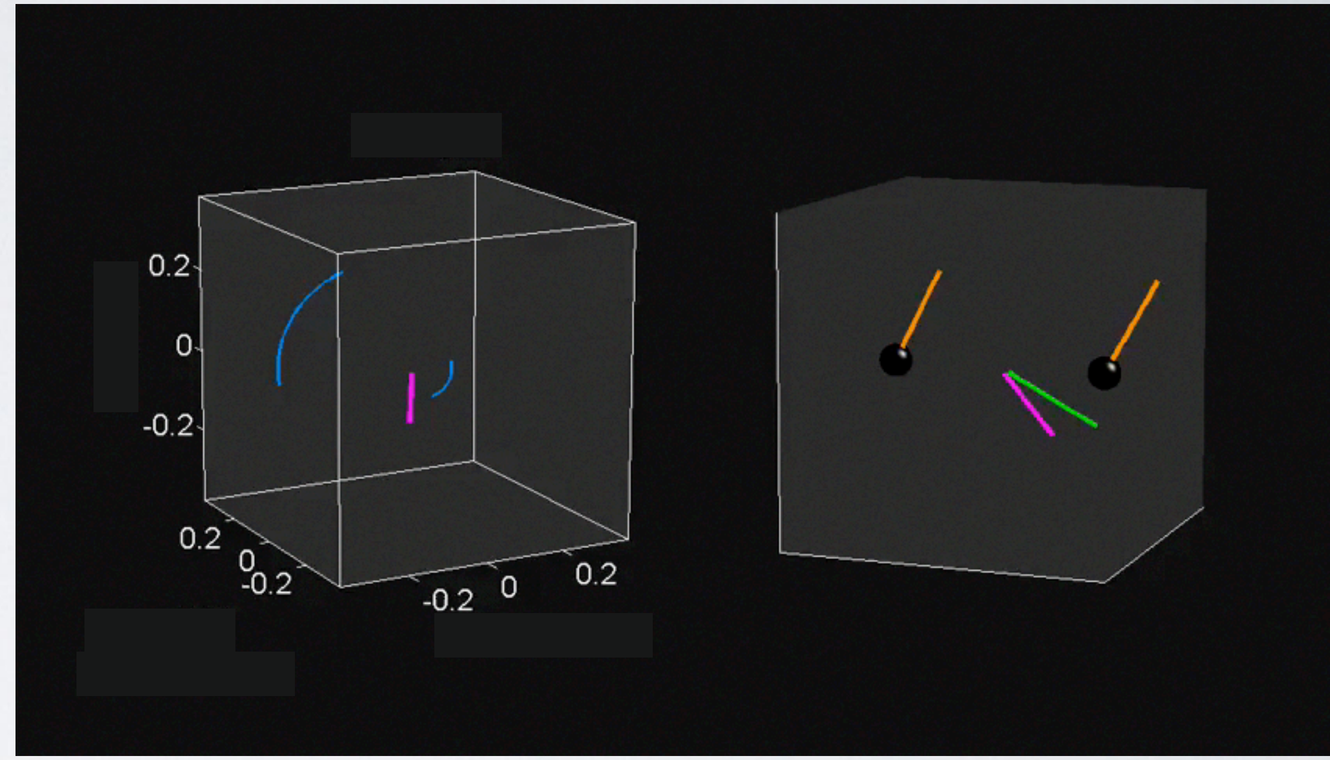
8.962 LECTURE 25

Going beyond symmetry: Wrap-up of post-Newtonian theory, survey of "strong-field" perturbation theory



Post-Newtonian spin precession due to "gravitomagnetic" effects in a binary system

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Recap: Foundation for iterating from weak to strong field begins by defining a spacetime variable

$$\mathbf{g}^{\mu\nu} \equiv \sqrt{-g}\, g^{\mu\nu}$$

(This variable is typically written as a "gothic" lower case g in much of the literature; the LaTeX interface I am using doesn't have that font.) In terms of this variable, the field equation takes the form

$$\partial_\mu \partial_\nu H^{\alpha\mu\beta\nu} = 16\pi G\left[(-g)T^{\alpha\beta}\right] + \Lambda^{\alpha\beta}$$

where $$H^{\alpha\mu\beta\nu} = \mathbf{g}^{\alpha\beta}\mathbf{g}^{\mu\nu} - \mathbf{g}^{\alpha\nu}\mathbf{g}^{\beta\mu}$$

To introduce the expansion, we define

$$h^{\alpha\beta} = \mathbf{g}^{\alpha\beta} - \eta^{\alpha\beta}$$

This is not necessarily small!

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Requiring this field to satisfy the gauge condition

$$\partial_\alpha h^{\alpha\beta} = 0$$

the field equation takes the form

$$\Box\, h^{\alpha\beta} = 16\pi G \tau^{\alpha\beta}\,, \quad \text{where} \quad \tau^{\alpha\beta} = (-g)T^{\alpha\beta} + \frac{\Lambda^{\alpha\beta}}{16\pi G}$$

This is the flat spacetime wave operator, so the solution is

$$h^{\alpha\beta}(\mathbf{x}, t) = -4G \int \frac{\tau^{\alpha\beta}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} d^3x'$$

At least formally, an exact solution for any kind of spacetime given any kind of source.

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A tad misleading: the quantity $\Lambda^{\alpha\beta}$ which appears here is a complicated nonlinear mess:

$$\Lambda^{\alpha\beta} = -h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} + \frac{1}{2}g^{\alpha\beta}g_{\mu\nu}\partial_\lambda h^{\mu\tau}\partial_\tau h^{\nu\lambda}$$
$$- g^{\alpha\mu}g_{\nu\tau}\partial_\lambda h^{\beta\tau}\partial_\mu h^{\nu\lambda} - g^{\beta\mu}g_{\nu\tau}\partial_\lambda h^{\alpha\tau}\partial_\mu h^{\nu\lambda} + g_{\mu\nu}g^{\lambda\tau}\partial_\lambda h^{\alpha\mu}\partial_\tau h^{\beta\nu}$$
$$+ \frac{1}{8}\left(2g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu}\right)\left(2g_{\lambda\tau}g_{\epsilon\pi} - g_{\tau\epsilon}g_{\lambda\pi}\right)\partial_\mu h^{\lambda\pi}\partial_\nu h^{\tau\epsilon}\,.$$

This can be reorganized into a form that makes it clear how different terms couple to one another:

$$\Lambda^{\alpha\beta} = N^{\alpha\beta}(h,h) + M^{\alpha\beta}(h,h,h) + L^{\alpha\beta}(h,h,h,h) + \ldots$$

Very well set up for an iterative solution:

$$h^{\alpha\beta} = \sum_{n=1}^{\infty} G^n h_n^{\alpha\beta}$$

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With this form, terms with $m < n$ act as sources at order n. Consider what the field equation looks like in a region with $T^{\alpha\beta} = 0$:

$$\Box h_1^{\alpha\beta} = 0$$

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$$\Box h_3^{\alpha\beta} = M^{\alpha\beta}(h_1, h_1, h_1) + N^{\alpha\beta}(h_1, h_2) + N^{\alpha\beta}(h_2, h_1)$$

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$$\Box h_4^{\alpha\beta} = L^{\alpha\beta}(h_1, h_1, h_1, h_1) + M^{\alpha\beta}(h_1, h_1, h_2) + M^{\alpha\beta}(h_1, h_2, h_1)$$
$$+ M^{\alpha\beta}(h_2, h_1, h_1) + N^{\alpha\beta}(h_2, h_2) + N^{\alpha\beta}(h_1, h_3) + N^{\alpha\beta}(h_3, h_1)$$

Motion in the spacetime we find

Examine the motion of "body 1" in spacetime we find. (Similar result describes "body 2," so binary is fully understood.

$$
\begin{aligned}
a_1^i &= -\frac{Gm_2 n_{12}^i}{r_{12}^2} \\
&+\frac{1}{c^2}\left\{ \left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \frac{Gm_2}{r_{12}^2}\left(\frac{3}{2}(n_{12}v_2)^2 - v_1^2 + 4(v_1 v_2) - 2v_2^2\right)\right] n_{12}^i \right. \\
&\left. + \frac{Gm_2}{r_{12}^2}\left(4(n_{12}v_1) - 3(n_{12}v_2)\right) v_{12}^i \right\}
\end{aligned}
$$

Leading term: Newtonian gravity.

Corrections due to leading non-linear term.

More corrections ...

$$+\frac{1}{c^4}\Bigg\{\bigg[-\frac{57G^3m_1^2m_2}{4r_{12}^4}-\frac{69G^3m_1m_2^2}{2r_{12}^4}-\frac{9G^3m_2^3}{r_{12}^4}$$

$$+\frac{Gm_2}{r_{12}^2}\bigg(-\frac{15}{8}(n_{12}v_2)^4+\frac{3}{2}(n_{12}v_2)^2v_1^2-6(n_{12}v_2)^2(v_1v_2)-2(v_1v_2)^2+\frac{9}{2}(n_{12}v_2)^2v_2^2$$

$$+4(v_1v_2)v_2^2-2v_2^4\bigg)$$

$$+\frac{G^2m_1m_2}{r_{12}^3}\left(\frac{39}{2}(n_{12}v_1)^2-39(n_{12}v_1)(n_{12}v_2)+\frac{17}{2}(n_{12}v_2)^2-\frac{15}{4}v_1^2-\frac{5}{2}(v_1v_2)+\frac{5}{4}v_2^2\right)$$

$$+\frac{G^2m_2^2}{r_{12}^3}\left(2(n_{12}v_1)^2-4(n_{12}v_1)(n_{12}v_2)-6(n_{12}v_2)^2-8(v_1v_2)+4v_2^2\right)\bigg]n_{12}^i$$

$$+\bigg[\frac{G^2m_2^2}{r_{12}^3}\left(-2(n_{12}v_1)-2(n_{12}v_2)\right)+\frac{G^2m_1m_2}{r_{12}^3}\left(-\frac{63}{4}(n_{12}v_1)+\frac{55}{4}(n_{12}v_2)\right)$$

$$+\frac{Gm_2}{r_{12}^2}\bigg(-6(n_{12}v_1)(n_{12}v_2)^2+\frac{9}{2}(n_{12}v_2)^3+(n_{12}v_2)v_1^2-4(n_{12}v_1)(v_1v_2)$$

$$+4(n_{12}v_2)(v_1v_2)+4(n_{12}v_1)v_2^2-5(n_{12}v_2)v_2^2\bigg)\bigg]v_{12}^i\Bigg\}$$

$$+\frac{1}{c^5}\Bigg\{\left[\frac{208G^3m_1m_2^2}{15r_{12}^4}(n_{12}v_{12})-\frac{24G^3m_1^2m_2}{5r_{12}^4}(n_{12}v_{12})+\frac{12G^2m_1m_2}{5r_{12}^3}(n_{12}v_{12})v_{12}^2\right]n_{12}^i$$

$$+\left[\frac{8G^3m_1^2m_2}{5r_{12}^4}-\frac{32G^3m_1m_2^2}{5r_{12}^4}-\frac{4G^2m_1m_2}{5r_{12}^3}v_{12}^2\right]v_{12}^i\Bigg\}$$

And some more.

$$+\frac{1}{c^6}\bigg\{\bigg[\frac{Gm_2}{r_{12}^2}\bigg(\frac{35}{16}(n_{12}v_2)^6-\frac{15}{8}(n_{12}v_2)^4v_1^2+\frac{15}{2}(n_{12}v_2)^4(v_1v_2)+3(n_{12}v_2)^2(v_1v_2)^2$$

$$-\frac{15}{2}(n_{12}v_2)^4v_2^2+\frac{3}{2}(n_{12}v_2)^2v_1^2v_2^2-12(n_{12}v_2)^2(v_1v_2)v_2^2-2(v_1v_2)^2v_2^2$$

$$+\frac{15}{2}(n_{12}v_2)^2v_2^4+4(v_1v_2)v_2^4-2v_2^6\bigg)$$

$$+\frac{G^2m_1m_2}{r_{12}^3}\bigg(-\frac{171}{8}(n_{12}v_1)^4+\frac{171}{2}(n_{12}v_1)^3(n_{12}v_2)-\frac{723}{4}(n_{12}v_1)^2(n_{12}v_2)^2$$

$$+\frac{383}{2}(n_{12}v_1)(n_{12}v_2)^3-\frac{455}{8}(n_{12}v_2)^4+\frac{229}{4}(n_{12}v_1)^2v_1^2$$

$$-\frac{205}{2}(n_{12}v_1)(n_{12}v_2)v_1^2+\frac{191}{4}(n_{12}v_2)^2v_1^2-\frac{91}{8}v_1^4-\frac{229}{2}(n_{12}v_1)^2(v_1v_2)$$

$$+244(n_{12}v_1)(n_{12}v_2)(v_1v_2)-\frac{225}{2}(n_{12}v_2)^2(v_1v_2)+\frac{91}{2}v_1^2(v_1v_2)$$

$$-\frac{177}{4}(v_1v_2)^2+\frac{229}{4}(n_{12}v_1)^2v_2^2-\frac{283}{2}(n_{12}v_1)(n_{12}v_2)v_2^2$$

$$+\frac{259}{4}(n_{12}v_2)^2v_2^2-\frac{91}{4}v_1^2v_2^2+43(v_1v_2)v_2^2-\frac{81}{8}v_2^4\bigg)$$

$$+\frac{G^2m_2^2}{r_{12}^3}\bigg(-6(n_{12}v_1)^2(n_{12}v_2)^2+12(n_{12}v_1)(n_{12}v_2)^3+6(n_{12}v_2)^4$$

$$+4(n_{12}v_1)(n_{12}v_2)(v_1v_2)+12(n_{12}v_2)^2(v_1v_2)+4(v_1v_2)^2$$

$$-4(n_{12}v_1)(n_{12}v_2)v_2^2-12(n_{12}v_2)^2v_2^2-8(v_1v_2)v_2^2+4v_2^4\bigg)$$

$$+\frac{G^3m_2^3}{r_{12}^4}\left(-(n_{12}v_1)^2+2(n_{12}v_1)(n_{12}v_2)+\frac{43}{2}(n_{12}v_2)^2+18(v_1v_2)-9v_2^2\right)$$

$$+\frac{G^3m_1m_2^2}{r_{12}^4}\bigg(\frac{415}{8}(n_{12}v_1)^2-\frac{375}{4}(n_{12}v_1)(n_{12}v_2)+\frac{1113}{8}(n_{12}v_2)^2-\frac{615}{64}(n_{12}v_{12})^2\pi^2$$

$$+18v_1^2+\frac{123}{64}\pi^2v_{12}^2+33(v_1v_2)-\frac{33}{2}v_2^2\bigg)$$

$$+\frac{G^3m_1^2m_2}{r_{12}^4}\bigg(-\frac{45887}{168}(n_{12}v_1)^2+\frac{24025}{42}(n_{12}v_1)(n_{12}v_2)-\frac{10469}{42}(n_{12}v_2)^2+\frac{48197}{840}v_1^2$$

$$-\frac{36227}{420}(v_1v_2)+\frac{36227}{840}v_2^2+110(n_{12}v_{12})^2\ln\left(\frac{r_{12}}{r'_1}\right)-22v_{12}^2\ln\left(\frac{r_{12}}{r'_1}\right)\bigg)$$

$$+\frac{16G^4m_2^4}{r_{12}^5}+\frac{G^4m_1^2m_2^2}{r_{12}^5}\left(175-\frac{41}{16}\pi^2\right)+\frac{G^4m_1^3m_2}{r_{12}^5}\left(-\frac{3187}{1260}+\frac{44}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right)$$

$$+\frac{G^4m_1m_2^3}{r_{12}^5}\left(\frac{110741}{630}-\frac{41}{16}\pi^2-\frac{44}{3}\ln\left(\frac{r_{12}}{r'_2}\right)\right)\bigg]n_{12}^i$$

$$+\bigg[\frac{Gm_2}{r_{12}^2}\bigg(\frac{15}{2}(n_{12}v_1)(n_{12}v_2)^4-\frac{45}{8}(n_{12}v_2)^5-\frac{3}{2}(n_{12}v_2)^3v_1^2+6(n_{12}v_1)(n_{12}v_2)^2(v_1v_2)$$

$$-6(n_{12}v_2)^3(v_1v_2)-2(n_{12}v_2)(v_1v_2)^2-12(n_{12}v_1)(n_{12}v_2)^2v_2^2+12(n_{12}v_2)^3v_2^2$$

$$+(n_{12}v_2)v_1^2v_2^2-4(n_{12}v_1)(v_1v_2)v_2^2+8(n_{12}v_2)(v_1v_2)v_2^2+4(n_{12}v_1)v_2^4$$

$$-7(n_{12}v_2)v_2^4\bigg)$$

$$+\frac{G^2m_2^2}{r_{12}^3}\bigg(-2(n_{12}v_1)^2(n_{12}v_2)+8(n_{12}v_1)(n_{12}v_2)^2+2(n_{12}v_2)^3+2(n_{12}v_1)(v_1v_2)$$

$$+4(n_{12}v_2)(v_1v_2)-2(n_{12}v_1)v_2^2-4(n_{12}v_2)v_2^2\bigg)$$

$$+\frac{G^2m_1m_2}{r_{12}^3}\bigg(-\frac{243}{4}(n_{12}v_1)^3+\frac{565}{4}(n_{12}v_1)^2(n_{12}v_2)-\frac{269}{4}(n_{12}v_1)(n_{12}v_2)^2$$

$$-\frac{95}{12}(n_{12}v_2)^3+\frac{207}{8}(n_{12}v_1)v_1^2-\frac{137}{8}(n_{12}v_2)v_1^2-36(n_{12}v_1)(v_1v_2)$$

$$+\frac{27}{4}(n_{12}v_2)(v_1v_2)+\frac{81}{8}(n_{12}v_1)v_2^2+\frac{83}{8}(n_{12}v_2)v_2^2\bigg)$$

$$+\frac{G^3m_2^3}{r_{12}^4}\left(4(n_{12}v_1)+5(n_{12}v_2)\right)$$

$$+\frac{G^3m_1m_2^2}{r_{12}^4}\left(-\frac{307}{8}(n_{12}v_1)+\frac{479}{8}(n_{12}v_2)+\frac{123}{32}(n_{12}v_{12})\pi^2\right)$$

$$+\frac{G^3m_1^2m_2}{r_{12}^4}\left(\frac{31397}{420}(n_{12}v_1)-\frac{36227}{420}(n_{12}v_2)-44(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\right)\bigg]v_{12}^i\bigg\}$$

$$+\frac{1}{c^7}\bigg\{\bigg[\frac{G^4m_1^3m_2}{r_{12}^5}\left(\frac{3992}{105}(n_{12}v_1)-\frac{4328}{105}(n_{12}v_2)\right)$$

$$+\frac{G^4m_1^2m_2^2}{r_{12}^6}\left(-\frac{13576}{105}(n_{12}v_1)+\frac{2872}{21}(n_{12}v_2)\right)-\frac{3172}{21}\frac{G^4m_1m_2^3}{r_{12}^6}(n_{12}v_{12})$$

$$+\frac{G^3m_1^2m_2}{r_{12}^4}\bigg(48(n_{12}v_1)^3-\frac{696}{5}(n_{12}v_1)^2(n_{12}v_2)+\frac{744}{5}(n_{12}v_1)(n_{12}v_2)^2-\frac{288}{5}(n_{12}v_2)^3$$

$$-\frac{4888}{105}(n_{12}v_1)v_1^2+\frac{5056}{105}(n_{12}v_2)v_1^2+\frac{2056}{21}(n_{12}v_1)(v_1v_2)$$

$$-\frac{2224}{21}(n_{12}v_2)(v_1v_2)-\frac{1028}{21}(n_{12}v_1)v_2^2+\frac{5812}{105}(n_{12}v_2)v_2^2\bigg)$$

$$+\frac{G^3m_1m_2^2}{r_{12}^4}\bigg(-\frac{582}{5}(n_{12}v_1)^3+\frac{1746}{5}(n_{12}v_1)^2(n_{12}v_2)-\frac{1954}{5}(n_{12}v_1)(n_{12}v_2)^2$$

$$+158(n_{12}v_2)^3+\frac{3568}{105}(n_{12}v_{12})v_1^2-\frac{2864}{35}(n_{12}v_1)(v_1v_2)$$

$$+\frac{10048}{105}(n_{12}v_2)(v_1v_2)+\frac{1432}{35}(n_{12}v_1)v_2^2-\frac{5752}{105}(n_{12}v_2)v_2^2\bigg)$$

$$+\frac{G^2m_1m_2}{r_{12}^3}\bigg(-56(n_{12}v_{12})^5+60(n_{12}v_1)^3v_{12}^2-180(n_{12}v_1)^2(n_{12}v_2)v_{12}^2$$

$$+174(n_{12}v_1)(n_{12}v_2)^2v_{12}^2-54(n_{12}v_2)^3v_{12}^2-\frac{246}{35}(n_{12}v_{12})v_1^4$$

$$+\frac{1068}{35}(n_{12}v_1)v_1^2(v_1v_2)-\frac{984}{35}(n_{12}v_2)v_1^2(v_1v_2)-\frac{1068}{35}(n_{12}v_1)(v_1v_2)^2$$

$$+\frac{180}{7}(n_{12}v_2)(v_1v_2)^2-\frac{534}{35}(n_{12}v_1)v_1^2v_2^2+\frac{90}{7}(n_{12}v_2)v_1^2v_2^2$$

$$+\frac{984}{35}(n_{12}v_1)(v_1v_2)v_2^2-\frac{732}{35}(n_{12}v_2)(v_1v_2)v_2^2-\frac{204}{35}(n_{12}v_1)v_2^4$$

$$+\frac{24}{7}(n_{12}v_2)v_2^4\bigg)\bigg]n_{12}^i$$

$$+\bigg[-\frac{184}{21}\frac{G^4m_1^3m_2}{r_{12}^5}+\frac{6224}{105}\frac{G^4m_1^2m_2^2}{r_{12}^6}+\frac{6388}{105}\frac{G^4m_1m_2^3}{r_{12}^6}$$

$$+\frac{G^3m_1^2m_2}{r_{12}^4}\bigg(\frac{52}{15}(n_{12}v_1)^2-\frac{56}{15}(n_{12}v_1)(n_{12}v_2)-\frac{44}{15}(n_{12}v_2)^2-\frac{132}{35}v_1^2+\frac{152}{35}(v_1v_2)$$

$$-\frac{48}{35}v_2^2\bigg)$$

$$+\frac{G^3m_1m_2^2}{r_{12}^4}\bigg(\frac{454}{15}(n_{12}v_1)^2-\frac{372}{5}(n_{12}v_1)(n_{12}v_2)+\frac{854}{15}(n_{12}v_2)^2-\frac{152}{21}v_1^2$$

$$+\frac{2864}{105}(v_1v_2)-\frac{1768}{105}v_2^2\bigg)$$

$$+\frac{G^2m_1m_2}{r_{12}^3}\bigg(60(n_{12}v_{12})^4-\frac{348}{5}(n_{12}v_1)^2v_{12}^2+\frac{684}{5}(n_{12}v_1)(n_{12}v_2)v_{12}^2$$

$$-66(n_{12}v_2)^2v_{12}^2+\frac{334}{35}v_1^4-\frac{1336}{35}v_1^2(v_1v_2)+\frac{1308}{35}(v_1v_2)^2+\frac{654}{35}v_1^2v_2^2$$

$$-\frac{1252}{35}(v_1v_2)v_2^2+\frac{292}{35}v_2^4\bigg)\bigg]v_{12}^i\bigg\}$$

$$+\mathcal{O}\left(\frac{1}{c^8}\right).$$

Details in the reference by Blanchet.

Effect on things we measure

On an earlier pset, we derived the flux of energy carried by GWs from a circular binary:

$$\mathcal{F} = \frac{32 G^4}{5} \frac{\mu^2 M^3}{R^5}$$

$$\mu = m_1 m_2 / M \;, \quad M = m_1 + m_2$$

$$R = \text{separation of masses}$$

Rewrite in terms of an observable:

$$\Omega = \sqrt{\frac{GM}{R^3}}$$

$$x \equiv (GM\Omega)^{1/3} \;, \quad \nu = \mu/M$$

Leading order form we derived becomes

$$\mathcal{F} = \frac{32}{5} \frac{1}{G} \nu^2 x^5$$

Effect on things we measure

Let's look at how this is changed by all those terms we calculate:

$$\mathcal{F} = \frac{32}{5}\frac{1}{G}\nu^2 x^5$$

Effect on things we measure

Let's look at how this is changed by all those terms we calculate:

$$\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x + 4\pi x^{3/2}$$
$$+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right) \pi x^{5/2}$$
$$+ \left[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_{\mathrm{E}} - \frac{856}{105}\ln(16\,x)\right.$$
$$\left. + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\right] x^3$$
$$+ \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right) \pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right) \Bigg\} .$$

New stuff: gravitomagnetism

Recall problem set #7 exercises: in linearized theory, found components of the spacetime arise from the *flow* of mass and energy:

$$ds^2 = -(1+2\Phi)dt^2 + \beta^i(dx^i dt + dt dx^i) + (1-2\Phi)(dx^2+dy^2+dz^2)$$

$$g_i \equiv \partial_i \Phi \ , \qquad H_i \equiv \epsilon_{ijk}\partial_j \beta_k$$

$$\partial_i g_i = -4\pi G\rho \ , \qquad \epsilon_{ijk}\partial_j g_k = 0$$

$$\partial_i H_i = 0 \ , \qquad \epsilon_{ijk}\partial_j H_k = -16\pi G(\rho v_i)$$

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$$g_i \equiv \partial_i \Phi \;, \qquad H_i \equiv \epsilon_{ijk}\partial_j \beta_k$$

Showed on pset that this leads to magnetic-like force law in these coordinates: Geodesic (free-fall) motion obeys

$$\frac{d^2x^i}{dt^2} = g_i + \epsilon_{ijk} v_j H_k$$

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$$ds^2 = -(1+2\Phi)dt^2 + \beta^i(dx^i dt + dt dx^i) + (1-2\Phi)(dx^2 + dy^2 + dz^2)$$

$$g_i \equiv \partial_i \Phi \ , \qquad H_i \equiv \epsilon_{ijk}\partial_j \beta_k$$

Not difficult to show that a spinning body precesses very similarly to how a magnetic dipole precesses in an external **B** field:

$$\frac{dS_i}{dt} = \frac{1}{2}\epsilon_{ijk} H_j S_k$$

New stuff: gravitomagnetism

Magnetic-like contribution to the spacetime drives magnetic-like precession of binary members' spins.

$$\frac{d\mathbf{S}_1}{dt} = \frac{1}{r^3}\left[\left(2+\frac{3}{2}\frac{m_2}{m_1}\right)\mu\sqrt{Mr}\hat{\mathbf{L}}\right]\times\mathbf{S}_1 + \frac{1}{r^3}\left[\frac{1}{2}\mathbf{S}_2 - \frac{3}{2}(\mathbf{S}_2\cdot\hat{\mathbf{L}})\hat{\mathbf{L}}\right]\times\mathbf{S}_1$$

$$\frac{d\mathbf{S}_2}{dt} = \frac{1}{r^3}\left[\left(2+\frac{3}{2}\frac{m_1}{m_2}\right)\mu\sqrt{Mr}\hat{\mathbf{L}}\right]\times\mathbf{S}_2 + \frac{1}{r^3}\left[\frac{1}{2}\mathbf{S}_1 - \frac{3}{2}(\mathbf{S}_1\cdot\hat{\mathbf{L}})\hat{\mathbf{L}}\right]\times\mathbf{S}_2$$

Orbital motion contribution.

Contribution from other body's spin

Leads to new forces, modifying the orbital acceleration felt by each body.

New stuff: gravitomagnetism

Magnetic-like contribution to the spacetime drives magnetic-like precession of binary members' spins.

$$\frac{d\mathbf{S}_1}{dt} = \frac{1}{r^3}\left[\left(2+\frac{3}{2}\frac{m_2}{m_1}\right)\mu\sqrt{Mr}\hat{\mathbf{L}}\right]\times\mathbf{S}_1 + \frac{1}{r^3}\left[\frac{1}{2}\mathbf{S}_2 - \frac{3}{2}(\mathbf{S}_2\cdot\hat{\mathbf{L}})\hat{\mathbf{L}}\right]\times\mathbf{S}_1$$

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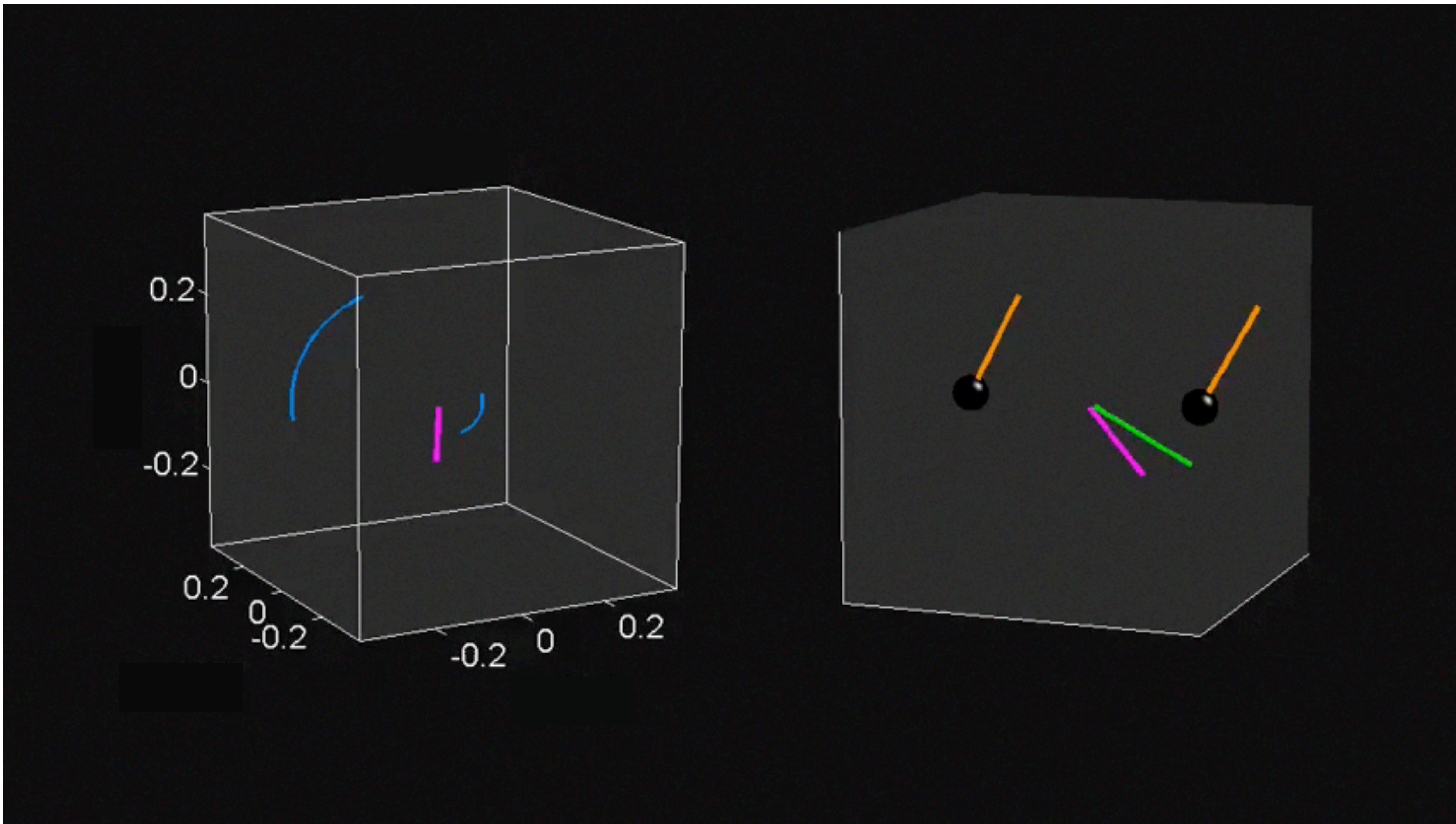
Angular momentum is *globally* conserved:

$\mathbf{J} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2 =$ constant

Orbital plane precesses to compensate for precession of the individual spins.

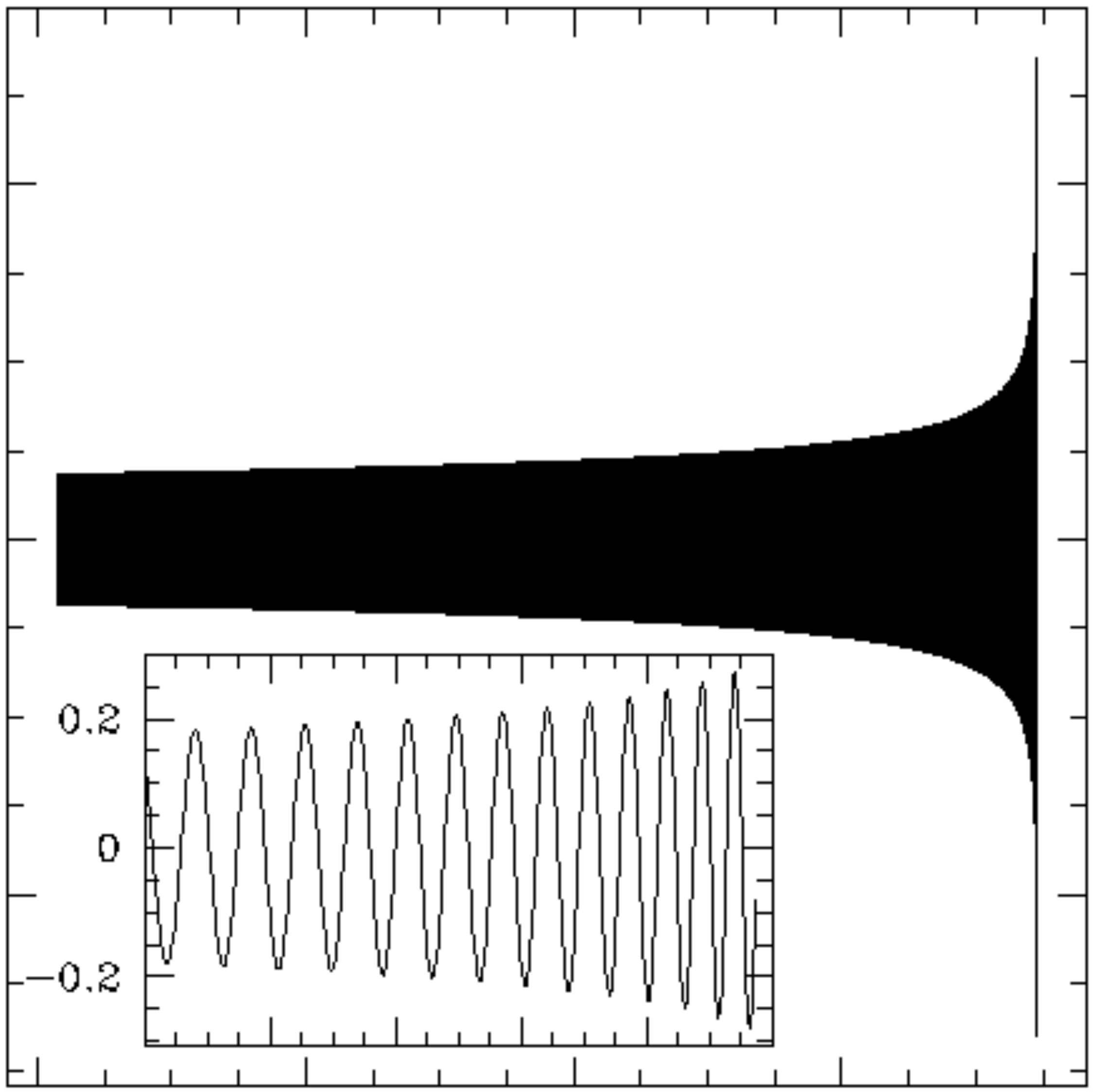
Impact on binary dynamics

Each body's spin enters dynamics … consistent with idea that all forms of energy gravitate.



Dynamics directly imprint GWs

Post-Newtonian descriptions directly "flavor" models that are used to measure GWs today.



Example 1:
Two non-spinning black holes.

Dynamics directly imprint GWs

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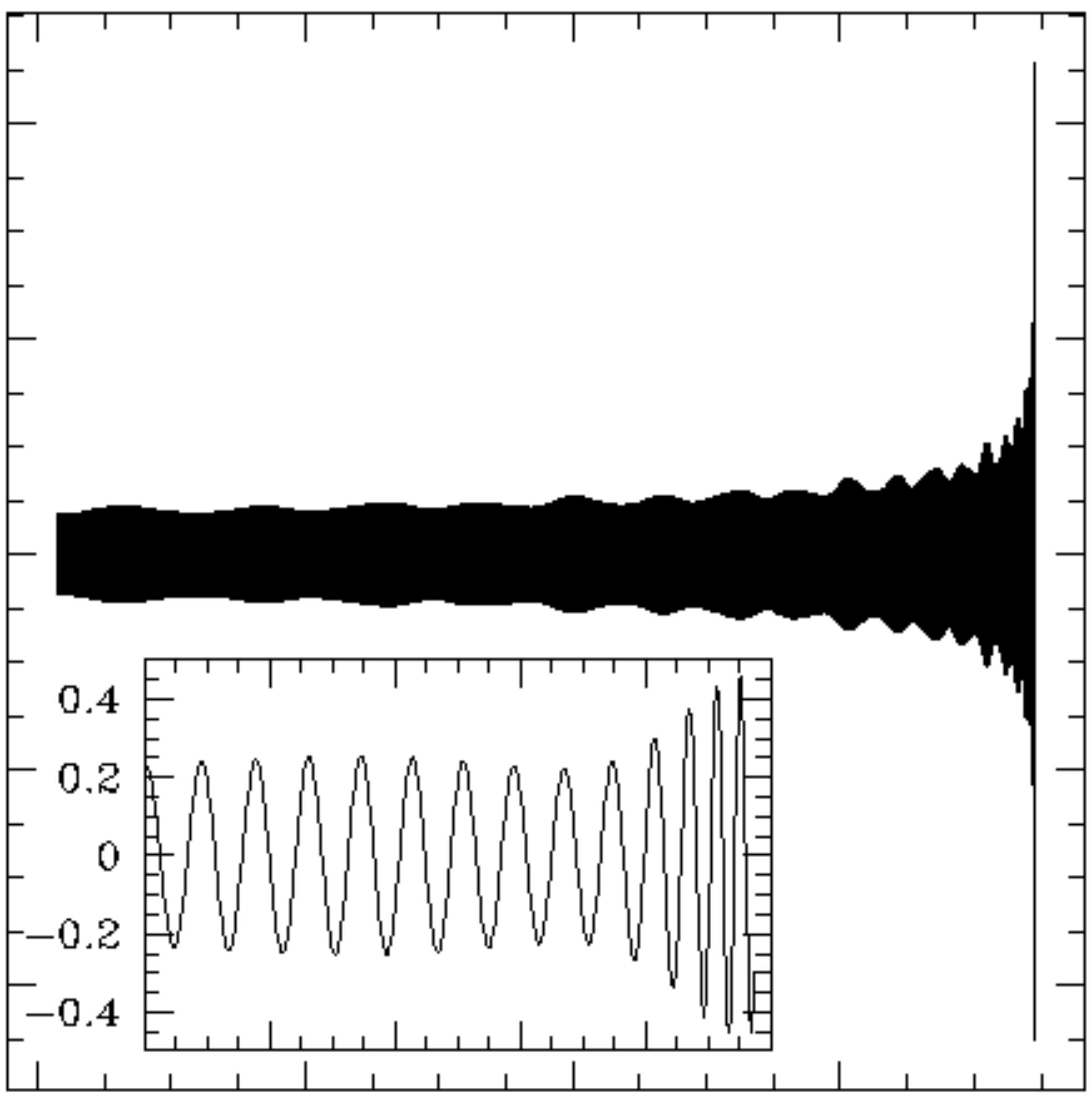

Example 2:
Two *rapidly* spinning black holes.

Results for odd parity

$$h_{00} = 0$$

$$h_{0i} \doteq H_0(t,r)\left[0, -\csc\theta\partial_\phi Y_{\ell m}, \sin\theta\partial_\theta Y_{\ell m}\right]$$

$$h_{ij} = H_1(t,r)\mathbf{e}^1_{ij} + H_2(t,r)\mathbf{e}^2_{ij}$$

See Rezzolla (gr-qc/0302025) for details and the explicit form of the basis tensors (which include first and second derivatives of Y_{lm}).

Can put $m = 0$ (axisymmetry); can set $H_2 = 0$ by choice of gauge.

The Regge-Wheeler equation

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} + \left(1 - \frac{2GM}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6GM}{r^3}\right] Q = 0$$

$$r_* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

$$Q \equiv \frac{H_1}{r}\left(1 - \frac{2GM}{r}\right)$$

$$\frac{\partial H_0}{\partial t} = \frac{\partial}{\partial r_*}(r_* Q)$$

As $r \to \infty$ and $r \to 2GM$, this simplifies:

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} = 0$$ so $Q \sim e^{i\omega(t \pm r_*)}$ in this limit.

Minus sign corresponds to outgoing wave packet; plus sign to ingoing.

The Regge-Wheeler equation

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Physics: Nothing can come out of the event horizon; nothing can come in from infinity.

The Regge-Wheeler equation

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} + \left(1 - \frac{2GM}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6GM}{r^3}\right] Q = 0$$

$$Q \equiv \frac{H_1}{r}\left(1 - \frac{2GM}{r}\right)$$

$$r_* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

$$\frac{\partial H_0}{\partial t} = \frac{\partial}{\partial r_*}(r_* Q)$$

Require

$$Q \sim e^{i\omega(t - r_*)} \qquad r \to \infty$$

$$Q \sim e^{i\omega(t + r_*)} \qquad r \to 2GM$$

The Regge-Wheeler equation

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} + \left(1 - \frac{2GM}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} - \frac{6GM}{r^3}\right] Q = 0$$

$$r_* = r + 2GM \ln\left(\frac{r}{2GM} - 1\right)$$

$$Q \equiv \frac{H_1}{r}\left(1 - \frac{2GM}{r}\right)$$

$$\frac{\partial H_0}{\partial t} = \frac{\partial}{\partial r_*}(r_* Q)$$

Require

$$Q \sim e^{i\omega(t-r_*)} \qquad r \to \infty$$

$$Q \sim e^{i\omega(t+r_*)} \qquad r \to 2GM$$

Turns into an eigenvalue problem: There exist special frequencies such that *Q* is purely ingoing on the horizon, purely outgoing at infinity.

Quasi-normal modes

Frequencies which do this depend on the angular index *l*, and have real and imaginary parts:

$$\omega = \omega_r + i\omega_i$$

Outgoing waves then have the form

$$Q \sim e^{-t/\tau} e^{i\omega_r (t - r_*)} \,, \qquad \tau = 1/\omega_i$$

Values of ω_r and τ in general need to be found numerically … l = 2 yields the longest lived modes:

$$\omega_r \simeq \frac{0.37}{GM} \quad \rightarrow \quad f_r \equiv \frac{\omega_r}{2\pi} = 240\,\mathrm{Hz}\left(\frac{50\,M_\odot}{M}\right)$$

$$\tau\omega_r \simeq 4$$

Vacuum; pick components

In vacuum, Ricci curvature vanishes; Riemann is equivalent to Weyl: $R_{\alpha\beta\gamma\delta} \rightarrow C_{\alpha\beta\gamma\delta}$.

Can also organize the components by projecting them onto a special set of basis vectors:

$$l^\alpha \doteq \frac{1}{\Delta}\left[(r^2+a^2), \Delta, 0, a\right]$$

(Tangent to outgoing null geodesics)

$$n^\alpha \doteq \frac{1}{2(r^2+a^2\cos^2\theta)}\left[(r^2+a^2), -\Delta, 0, a\right]$$

(Tangent to ingoing null)

$$m^\alpha \doteq \frac{1}{\sqrt{2}(r+ia\cos\theta)}\left[ia\sin\theta, 0, 1, i\csc\theta\right]$$

(This plus complex conjugate cover the angular degrees of freedom)

Complex Weyl scalars

10 degrees of freedom in the Weyl curvature are given by the following 5 complex numbers:

$$\Psi_0 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta l^\gamma m^\delta \quad \Psi_4 = -C_{\alpha\beta\gamma\delta} n^\alpha \bar{m}^\beta n^\gamma \bar{m}^\delta$$

$$\Psi_1 = -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta l^\gamma m^\delta \quad \Psi_3 = -C_{\alpha\beta\gamma\delta} l^\alpha n^\beta \bar{m}^\gamma n^\delta$$

$$\Psi_2 = -C_{\alpha\beta\gamma\delta} l^\alpha m^\beta \bar{m}^\gamma n^\delta$$

Introduce perturbative expansion for these quantities:

$$\Psi_n = \Psi_n^{\mathrm{B}} + \delta\Psi_n$$

Remake the Riemann wave equation in terms of $\delta\Psi_n$

Done by Teukolsky in 1973.

Result 1: Ringing modes with spin

Focus on Ψ_4. The resulting equation describes radiation far from the perturbed black hole. The equation that results turns out to separate:

$$\Psi_4 = \frac{1}{(r - ia\cos\theta)^4} \int d\omega \sum_{\ell m} R_{\ell m\omega}(r) S_{\ell m\omega}(\theta) e^{im\phi} e^{-i\omega t}$$

Rather simple equations govern the behavior of *R* and *S*.

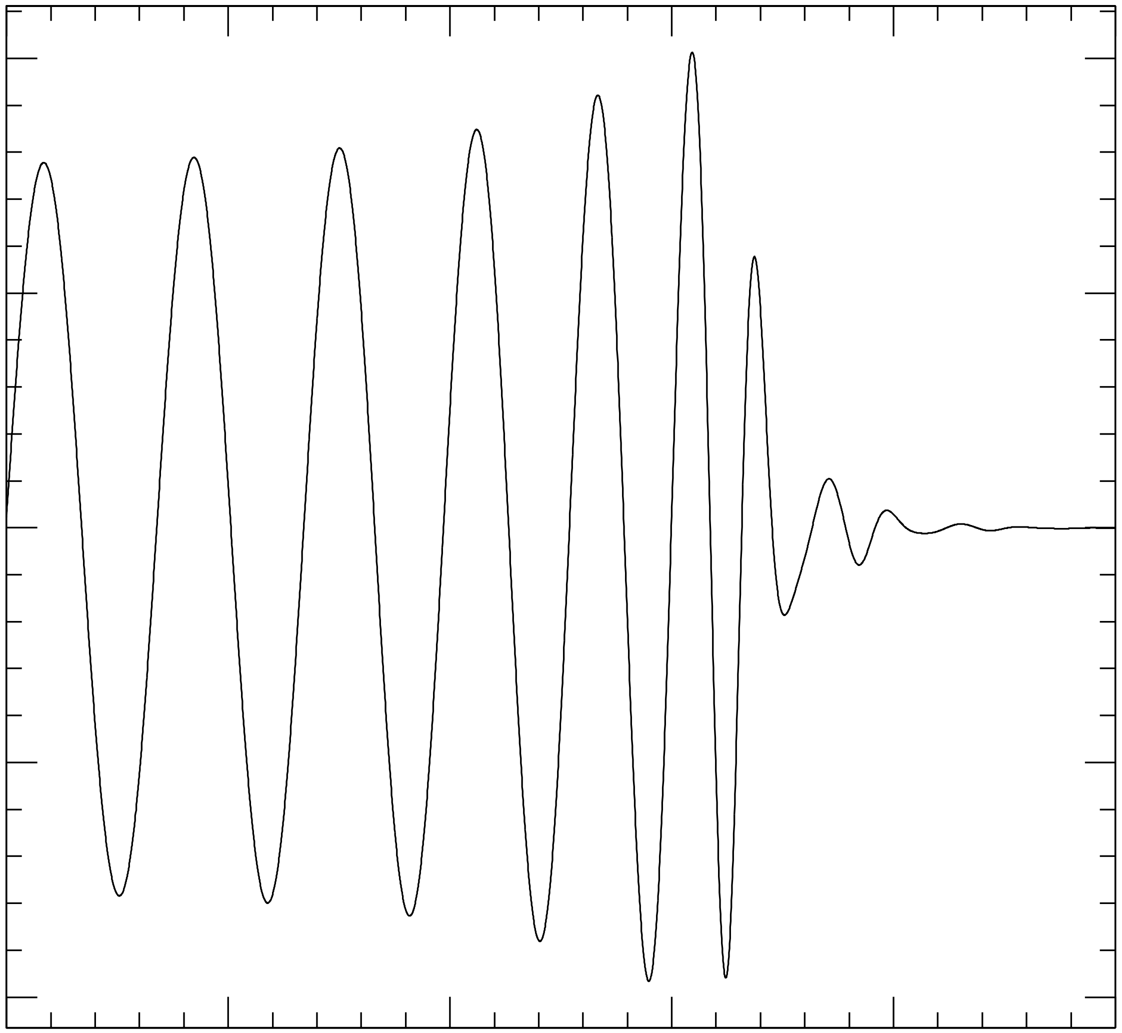
Again find special frequencies for which Ψ_4 is purely ingoing boundary at horizon, outgoing at infinity:

$$\Psi_4(r \to \infty) \sim e^{-t/\tau_{\ell m}} e^{i\omega_{\ell m} t}$$

$$\omega_{22} \approx \frac{1}{GM}\left[1 - 0.63(1-a)^{0.7}\right] \qquad \tau_{22}\omega_{22} \approx 4(1-a)^{-0.45}$$

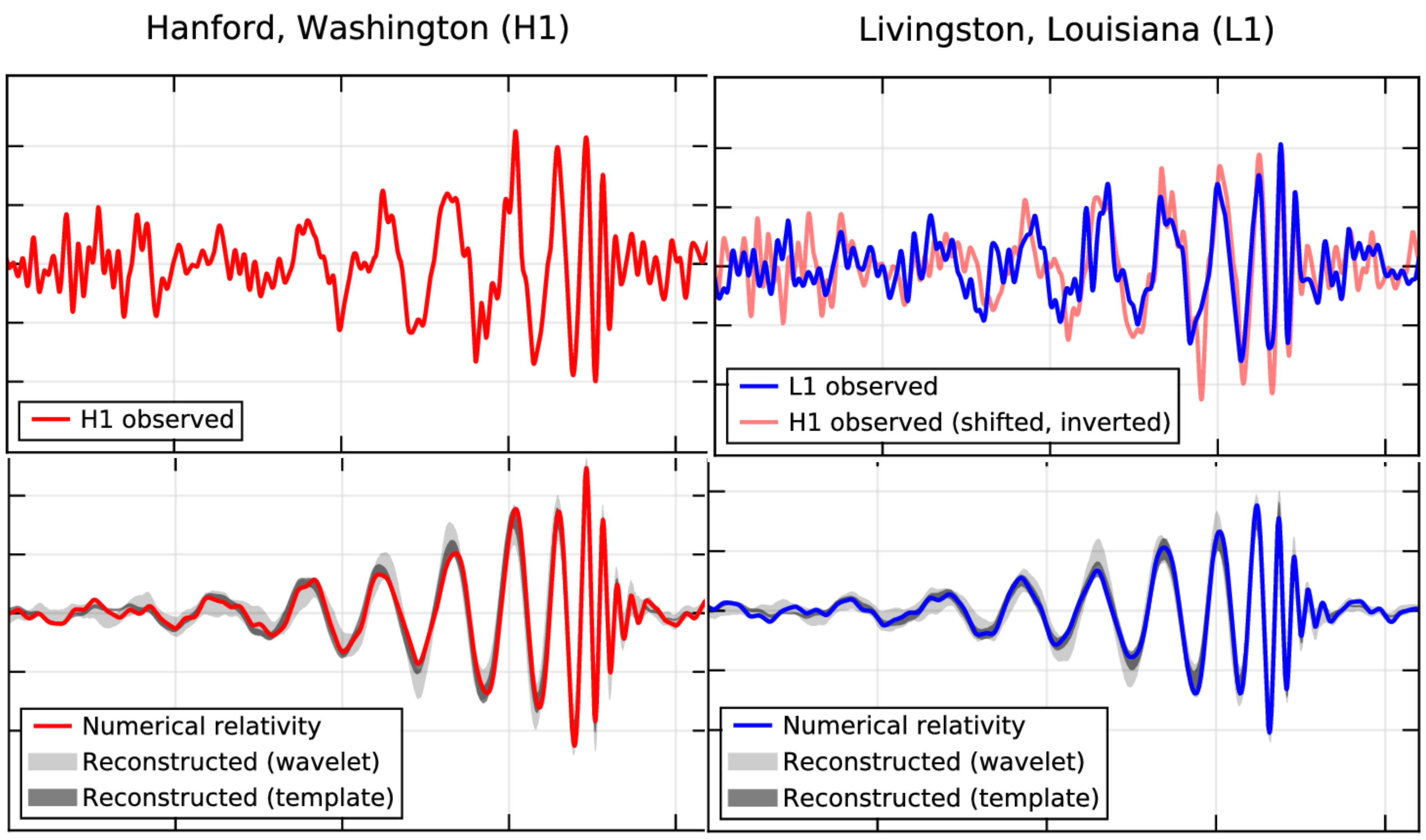
Example of ringdown

Example for a black hole with spin a = 0.8M



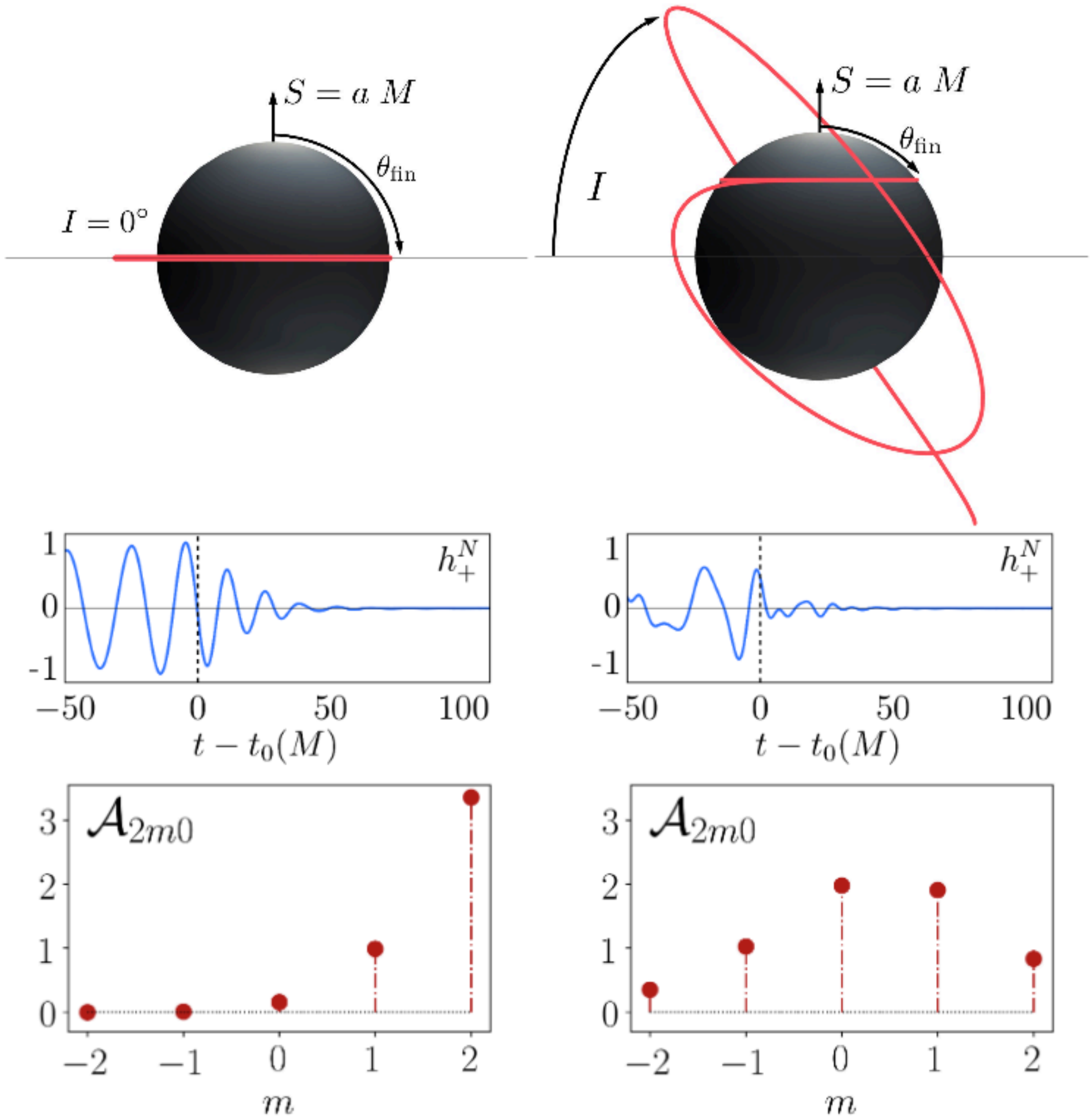
Frequency and damping time determined by — and thus encode — the mass and spin of the final black hole.

First event shows some of the best evidence of this structure



Last few cycles of GW150914 are consistent with this structure, for a final black hole with $a/M = 0.7$

More events will more fully explore parameter space



Example: work (arXiv:1901.05900; S.A. Hughes, A. Apte, G. Khanna, H. Lim), showing how the spectrum of mode excitation depends on the geometry of the final plunge.

With source: model binaries in which smaller body perturbs the spacetime of the larger one