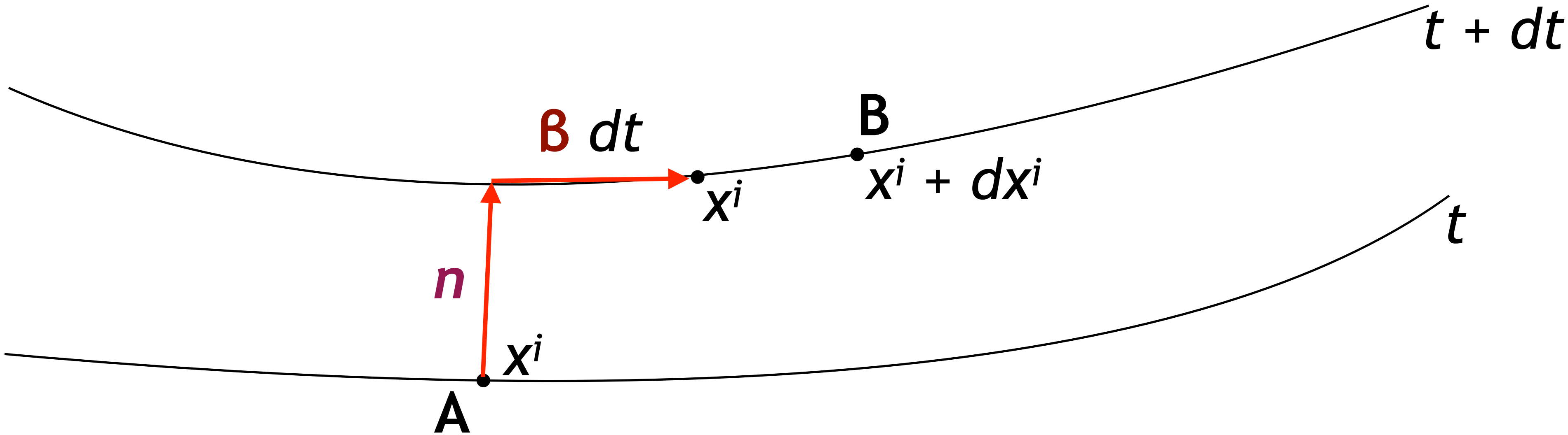
Split spacetime into space & time



Event **A**: (t, x^i) Event **B**: $(t + dt, x^i + dx^i)$

n is normal to "time slice." Proper time experienced by an observer who moves along n^i from t to $t + dt$ is $d\tau = \alpha\, dt$ … function α, the *lapse*, converts coordinate interval to proper interval for a "normal" observer.

Lapse lets us run time at different rates in different parts of our spacetime.

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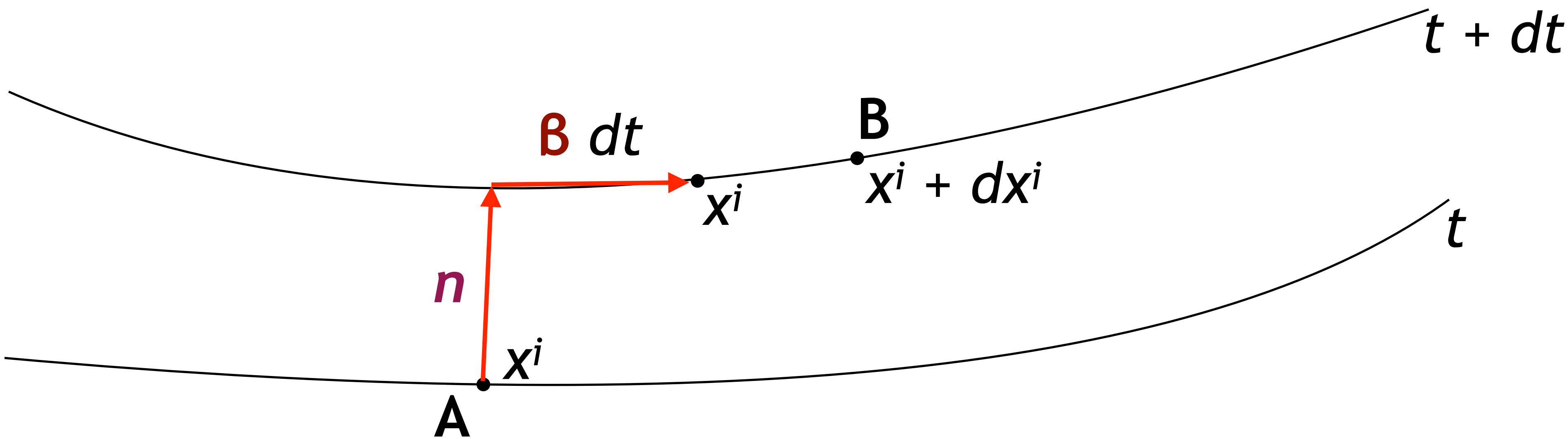
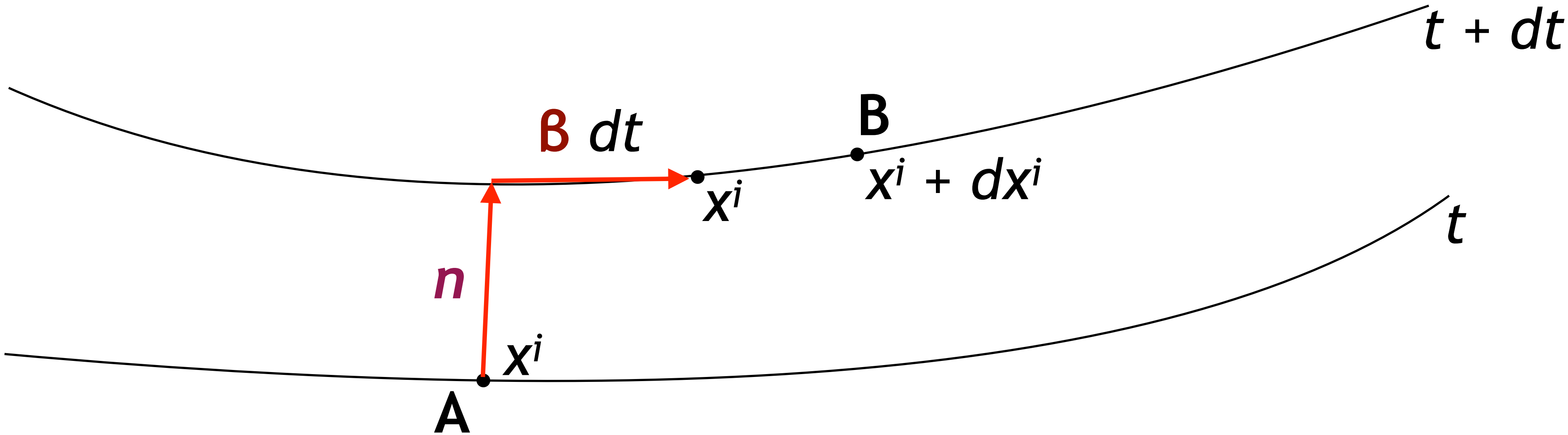


Event **A**: (t, x^i) Event **B**: $(t + dt, x^i + dx^i)$

$\beta^i\, dt$ is coordinate displacement of x^i in slice $t + dt$ from x^i in slice t. Called the "shift"; reflects freedom to slide spatial coordinates around in each timeslice.

The lapse α and the shift β^i generalize the notion of "gauge" freedom to a generic situation.

Split spacetime into space & time



Event **A**: (t, x^i) Event **B**: $(t + dt, x^i + dx^i)$

Total spacetime distance between events **A** and **B**:

$$ds^2 = -\alpha^2 dt^2 + g_{ij}(dx^i + \beta^i\, dt)(dx^j + \beta^j\, dt)$$

Some more careful definitions

Take spacetime manifold, "foliate" it with level surfaces of some scalar function t. Define a 1-form

$$\Omega_a = \nabla_a t \qquad \text{with norm} \qquad g^{ab}\Omega_a\Omega_b = -\frac{1}{\alpha^2}$$

Normalize this: $$\omega_a = \alpha\Omega_a$$

Define the corresponding vector: $$n^a = -g^{ab}\omega_a$$

n^a is the future-directed normal to the level surface of constant t. Not hard to show that $n^a n_a = -1$, can be regarded as the 4-velocity of a particular observer.

Auxiliary definition: $$t^a = \alpha n^a + \beta^a$$

β^a gives gauge freedom: can slide spatial coordinates around on each slice as we wish or need.

Some more careful definitions

Using this, define tensor that projects orthogonal to n^a:

$$\gamma_{ab} = g_{ab} + n_a n_b$$

This is tensor describes space geometry in the constant *t* "slice" … it *is* the metric for the slice's 3-geometry.

Any tensor in a slice is then given by contracting:

$$[A^a{}_b]_{\text{in slice}} = \gamma^a{}_c \gamma^b{}_d A^c{}_d$$

Particularly useful: covariant derivative in slice:

$$\left[D_a A^b\right]_{\text{in slice}} = \gamma^c{}_a \gamma^b{}_d \nabla_c A^d$$

Can show that $D_a \gamma_{bc} = 0$ … allows us to define Christoffel symbols in slice, write usual covariant derivative formula in any time slice.

Curvature

Last thing we need to do is develop the curvature of spacetime in this language. Two pieces:

1. *Intrinsic*: The curvature in a particular time slice. Just use γ_{ab}, develop Riemann as usual.

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Last thing we need to do is develop the curvature of spacetime in this language. Two pieces:

1. *Intrinsic*: The curvature in a particular time slice. Just use γ_{ab}, develop Riemann as usual.

2. *Extrinsic*: Curvature due to how each time slice is "embedded" in the 4-dimensional geometry.

This last notion of curvature is related to the expansion or divergence of the normal vectors.

Define: Expansion $\theta_{ab} = \gamma^c{}_a \gamma^d{}_b \nabla_c n_d$

Flip sign to be in accord with usual notions of curvature

The “extrinsic” curvature is then defined as

$$K_{ab} = -\gamma^c{}_a \gamma^d{}_b \nabla_c n_d$$

With some manipulation, can show that this is simply related to the *Lie derivative* of the spatial metric:

$$K_{ab} = -\frac{1}{2}\mathcal{L}_{\vec{n}}\gamma_{ab}$$

Note that this is really a *connection coefficient*: single derivative of the metric, connecting the geometry of a slice at “time” t to a slice at “time” $t + \delta t$.

Now, "just" need to project the Einstein field equations.

1. Time-time piece.

$$n^a n^b \, {}^{(4)}G_{ab} = 8\pi G T_{ab} n^a n^b \equiv 8\pi G \rho$$

This becomes

$$R + K^2 - K_{ab}K^{ab} = 16\pi G \rho$$

Known as the "Hamiltonian constraint." Relates geometry in a particular slice to the energy density in that slice as measured by the observer whose 4-velocity is n^a.

Note: R is Ricci scalar in a spatial slice.

Now, "just" need to project the Einstein field equations.

2. Time-space piece.

$$\gamma^a{}_c n^b G_{ab} = 8\pi G T_{ab} \gamma^a{}_c n^b \equiv -8\pi G j_c$$

This becomes

$$D_b K^b{}_a - D_a K = 8\pi G j_a$$

Known as the "Momentum constraint." Relates geometry in a particular slice to the momentum density in that slice as measured by the observer whose 4-velocity is n^a.

Now, "just" need to project the Einstein field equations.

3. Space-space piece.

$$\gamma^a{}_c \gamma^b{}_d G_{ab} = 8\pi G T_{ab} \gamma^a{}_c \gamma^b{}_d \equiv 8\pi G S_{cd}$$

This becomes

$$\mathcal{L}_{\vec{t}} K_{ab} = -D_a D_b \alpha + \alpha (R_{ab} - 2K_{ac} K^c{}_b + K K_{ab}) - 8\pi G \alpha \left[S_{ab} - \frac{1}{2} \gamma_{ab} (S - \rho) \right] + \mathcal{L}_{\vec{\beta}} K_{ab}$$

Known as the "Evolution equation." Tells us how geometry evolves from time slice to time slice.

Now, “just” need to project the Einstein field equations.

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$$\gamma^a{}_c \gamma^b{}_d G_{ab} = 8\pi G T_{ab} \gamma^a{}_c \gamma^b{}_d \equiv 8\pi G S_{cd}$$

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$$- 8\pi G \alpha \left[S_{ab} - \frac{1}{2} \gamma_{ab} (S - \rho) \right] + \mathcal{L}_{\vec{\beta}} K_{ab}$$

Once we’ve done all this, if we pick our coordinates well, the indices (*ab*) become (*ij*): non-spatial components are zero thanks to the projection tensors.

Now, "just" need to project the Einstein field equations.

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$$\begin{aligned} \mathcal{L}_{\vec{t}} K_{ab} = & -D_a D_b \alpha + \alpha(R_{ab} - 2K_{ac}K^c{}_b + K K_{ab}) \\ & - 8\pi G \alpha \left[S_{ab} - \frac{1}{2}\gamma_{ab}(S - \rho) \right] + \mathcal{L}_{\vec{\beta}} K_{ab} \end{aligned}$$

This split of Einstein is known as the "ADM" framework (for Arnowitt, Deser, and Misner). Originally developed in an attempt at developing quantum gravity, but application to classical gravity has proven very fruitful.

Theorem: Start with a slice that satisfies constraints; evolve; slice will continue to satisfy constraints.

Analogy to Maxwell: Constraints similar to divergence equations; evolution is similar to curl equations.

$$\nabla \cdot \mathbf{E} = 4\pi\rho_Q \qquad \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - 4\pi\mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

If (***E***, ***B***) obey divergence equations at an initial time, and you evolve forward in time with curl equations, then (***E***, ***B***) satisfy divergence equations at any later time.

Recipe

1. Pick coordinates. Amounts to coming up with a way of choosing lapse α and shift β^a.

2. Pick initial spacetime geometry. **Highly** nontrivial: Need to make γ_{ab} and K_{ab} that describe the situation you want to study, subject to the Hamiltonian and momentum constraints. Example: Two objects in a binary orbit. Might want to imagine they are enough apart that the post-Newtonian expansion describes them.

3. Evolve. ***If*** all is set up correctly, GR should just do its thing. For example, it should "automagically" respect ingoing boundary condition at event horizons. (Whose locations we cannot know in a dynamical spacetime until the entire calculation is completed.)

Typical result for several decades: **Catastrophic failure.**

Major reason: “Constraint violating modes”

Initial data (by construction) satisfies constraints … **up to some level of precision.** Numerical noise/roundoff error will introduce “pollution” that violates constraints.

Thanks to nonlinear character of evolution equation, “pollution” will typically *grow* as spacetime evolves. Get solution dominated by nonphysical data.

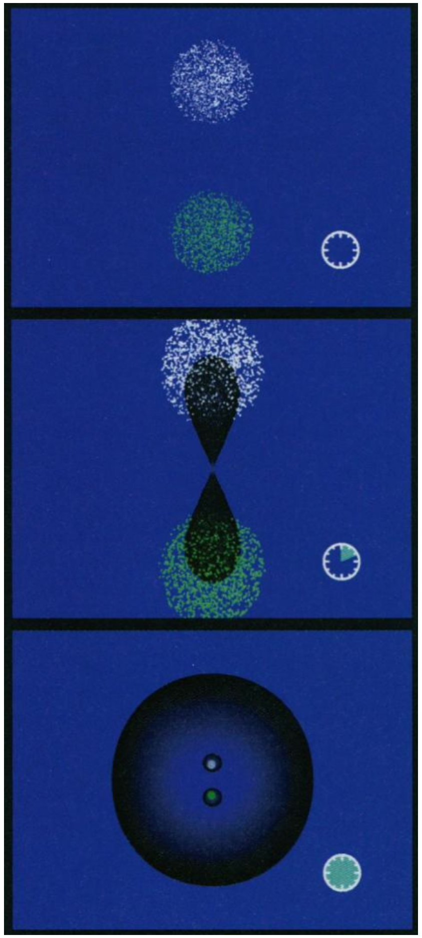
In principle can fix this by re-solving constraints … but when you do this, your system may not correspond to the system that you started with.

Brief history

First numerical solutions attempted in 1970s for highly symmetric situations. By 1990s, people were good at "2+1" problems (2 space, 1 time — e.g., axial symmetry). Example: Head-on collisions of two black holes.

Example results: two dust balls that, each collapse to form black holes, in a head-on collision. Horizon starts out disjoint and highly distorted, settles down to Schwarzschild solution at late times.

(Reference: Hughes et al, PRD 1994)



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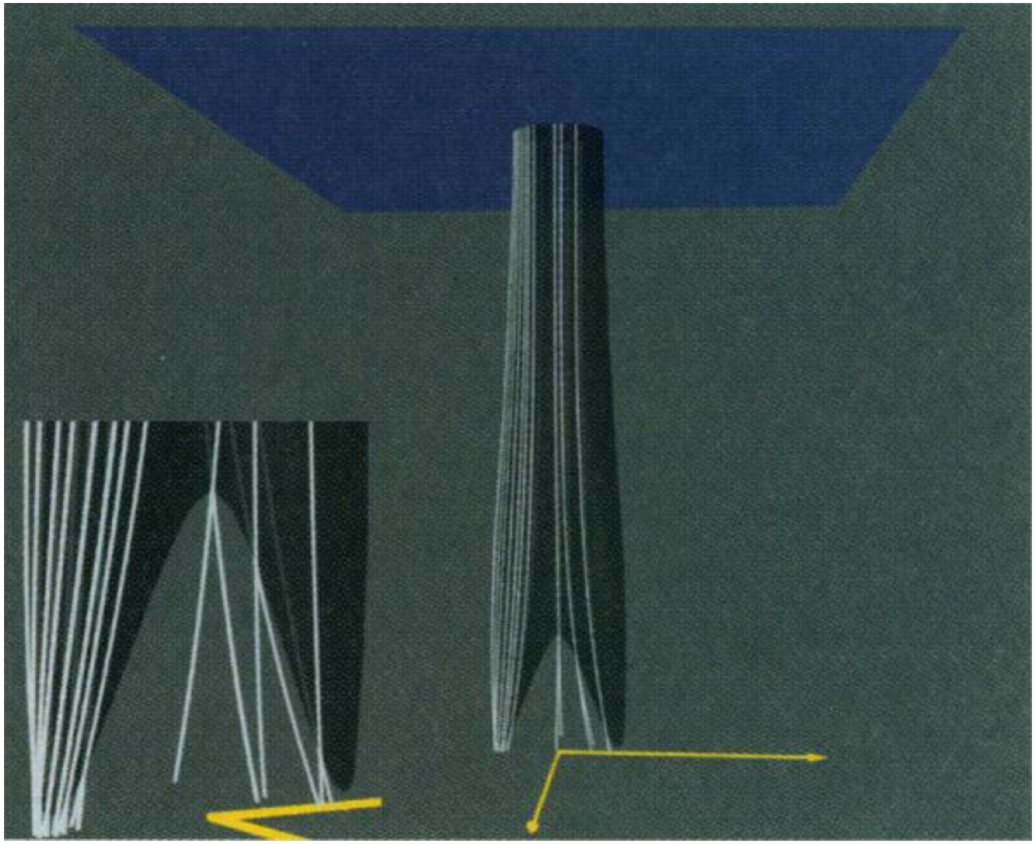

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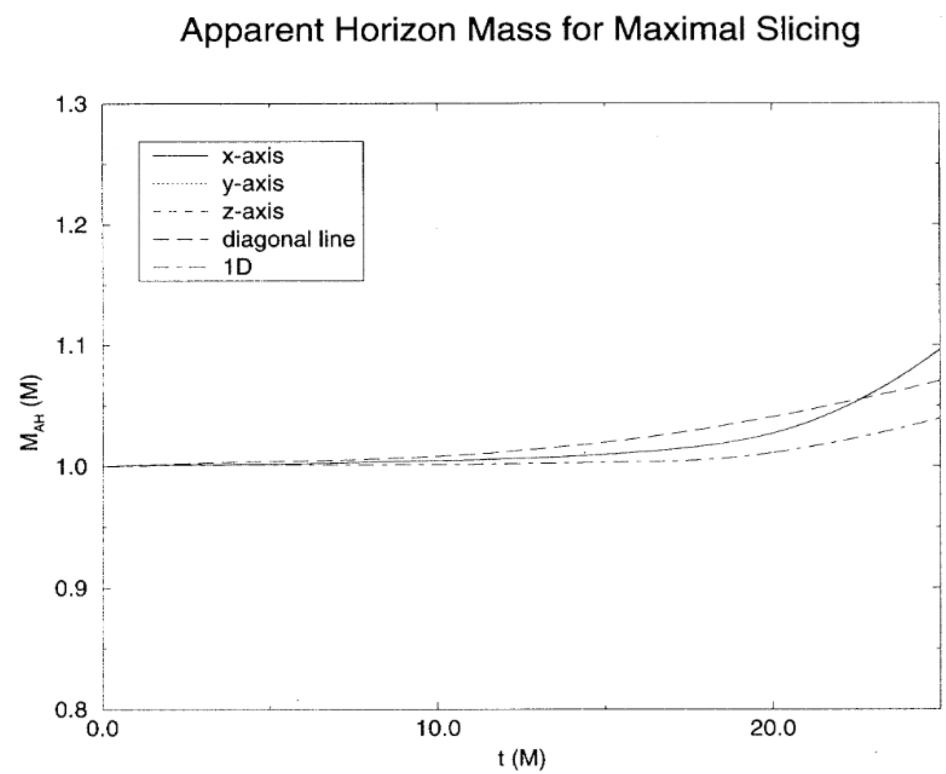
Full "3+1" much harder: if the equations you solve don't "know" about symmetries, many more instabilities can develop.

Example: Anninos et al 1995 (arXiv:gr-qc/9503025).

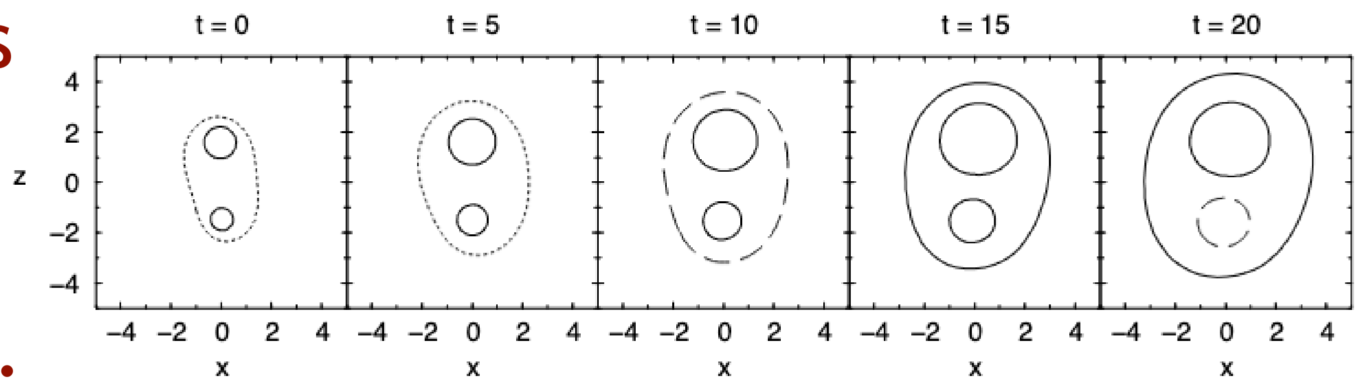


Code runs until *t* ~ 50*GM* then crashes!

Brief history

First attempt at evolving initial data that looked like orbiting black holes: Brügmann 1997 (arXiv:gr-qc/9708035). Black holes orbited for about 20*GM* before the code crashed (roughly 1/4 of an orbit).

Fig 8 of Brügmann's paper, showing an approximation for the horizon vs time.



Computations were done in a "co-rotating frame," so the black holes appear to sit still while the rest of the universe whirls around them.

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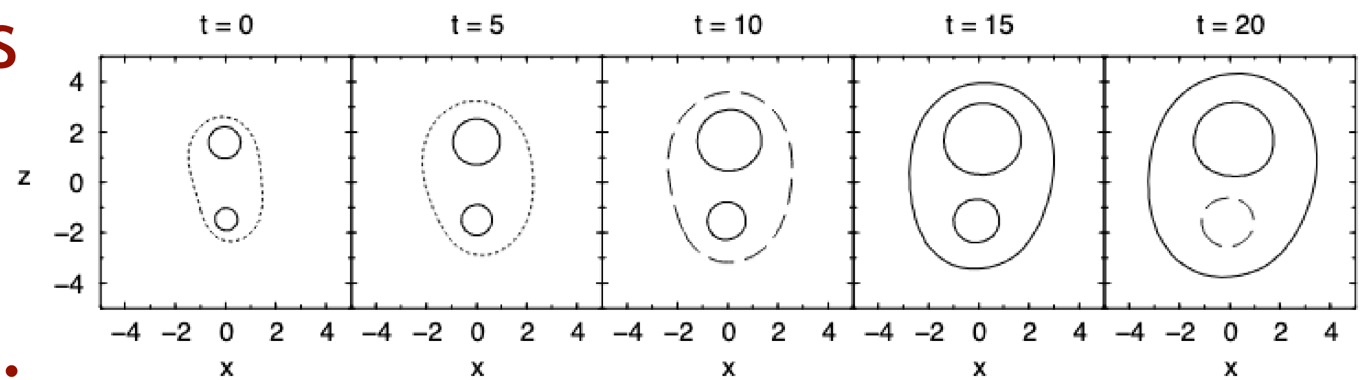


Bigwigs in the field typically called this problem the "Holy Grail" of general relativity.

(When one pointed out that literary "Holy Grail" quests did not end well for the questors, one did not make friends.)

Brief history

Stunning announcement by Frans Pretorius in 2005: A new formulation of the field equations developed using “generalized harmonic coordinates” makes it possible to compute binary black hole mergers.

Formulation sufficiently different from what “everyone” had been doing that community wondered … must we need start over from scratch with ***everything?***

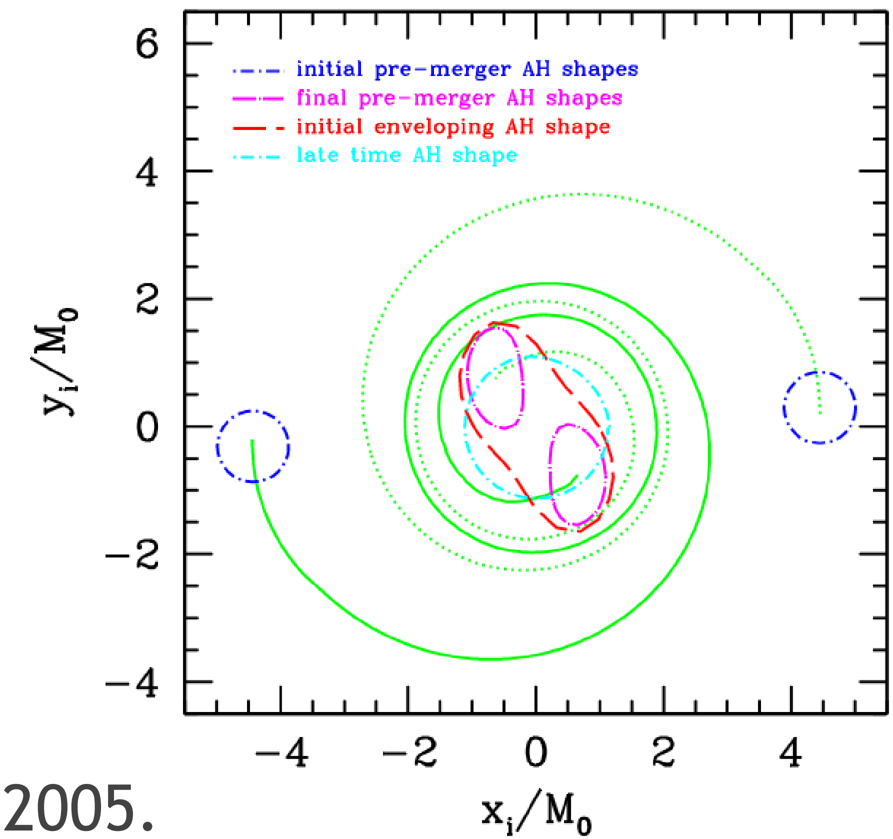


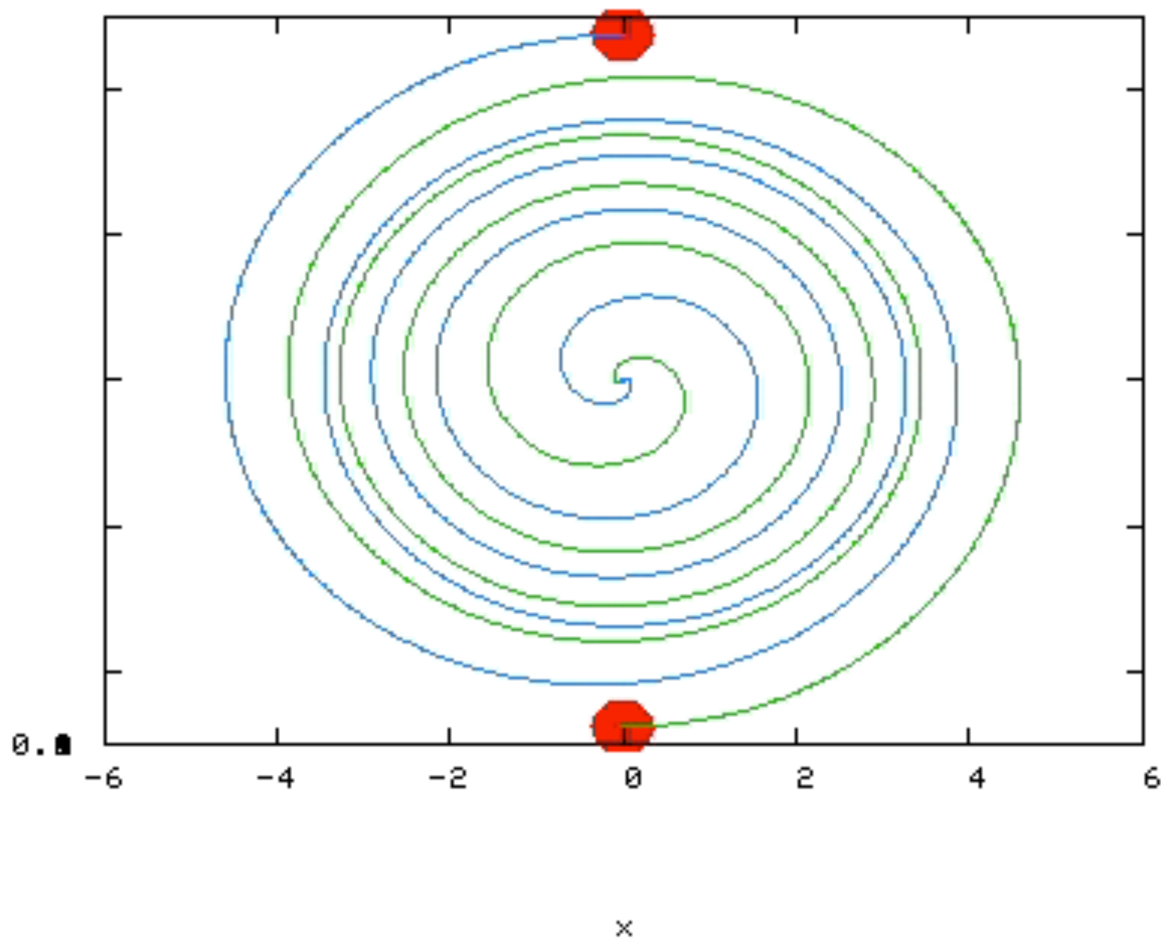
Figure from Pretorius PRD 2006; announcement was a PRL published in 2005.

No.

Several groups were able to adapt their codes (in part, using tricks learned from Pretorius), and before long "everyone" could do this.

This example by Baker et al, a group at Goddard Space Flight Center (NASA research facility).

When asked why the run terminated, their answer was "we got bored watching the black hole just sit there."



What changed?

Several ingredients proved really helpful:

1. A tweak to the representation of the "fundamental" fields:

$$\tilde{\gamma}_{ij} = e^{-4\phi}\gamma_{ij} \quad \text{where} \quad e^{12\phi} = \det(\gamma_{ij})$$

$$A_{ij} = K_{ij} - \frac{1}{3}\gamma_{ij}K\ ; \quad \tilde{A}_{ij} = e^{-4\phi}A_{ij}$$

Then rewrite the constraints and the evolution equations to use this "tweaked" form (see Baumgarte and Shapiro 1998).

This "tweaked" form separates "transverse" degrees of freedom ($\tilde{\gamma}_{ij}, \tilde{A}_{ij}$), often associated with quantities that propagate across spacetime (i.e., radiation), from "Coulomb" degrees of freedom (ϕ, K), which are associated with the geometry of an isolated source (like a star or black hole).

What changed?

Several ingredients proved really helpful:

2. Introduce a new field

$$\tilde{\Gamma}^i = \tilde{\gamma}^{jk}\tilde{\Gamma}^i_{jk}$$

Terms involving Ricci curvature can be rewritten in terms of this variable; has "nice" characteristics (purely elliptic operator acting on $\tilde{\Gamma}^i$ determines curvature).

Need to develop an evolution equation for this, but not difficult to do so once you have written out this quantity.

What changed?

Several ingredients proved really helpful:

3. Use constraint equation to simplify evolution equation.

Example: In these new variables, the evolution equation for *K* is given by

$$\frac{dK}{dt} = -\gamma^{ij} D_i D_j \alpha + \alpha R + \alpha \left(\tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} K^2 \right) - 4\pi G \left(3\rho - S \right)$$

The Hamiltonian constraint relates *R* to several of these quantities:

$$R + K^2 - \left(\tilde{A}_{ij} \tilde{A}^{ij} + \frac{1}{3} K^2 \right) = 16\pi G \rho$$

Evolution equation for $\tilde{\Gamma}^i$ can be similarly simplified by using the momentum constraint. Including these modifications greatly suppresses the growth of constraint-violating modes.

What changed?

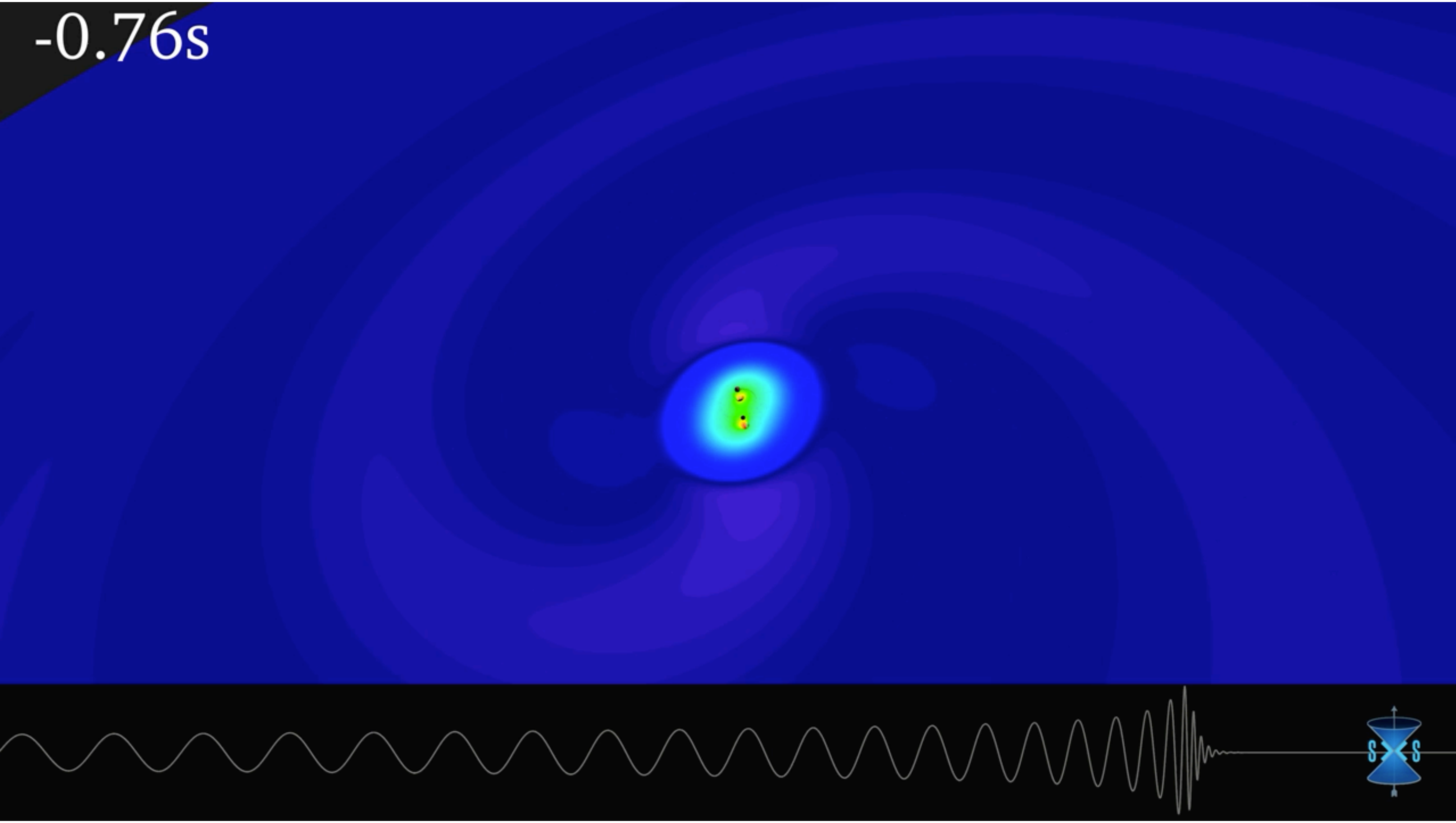
Several ingredients proved really helpful:

4. Smart "gauge" choices.

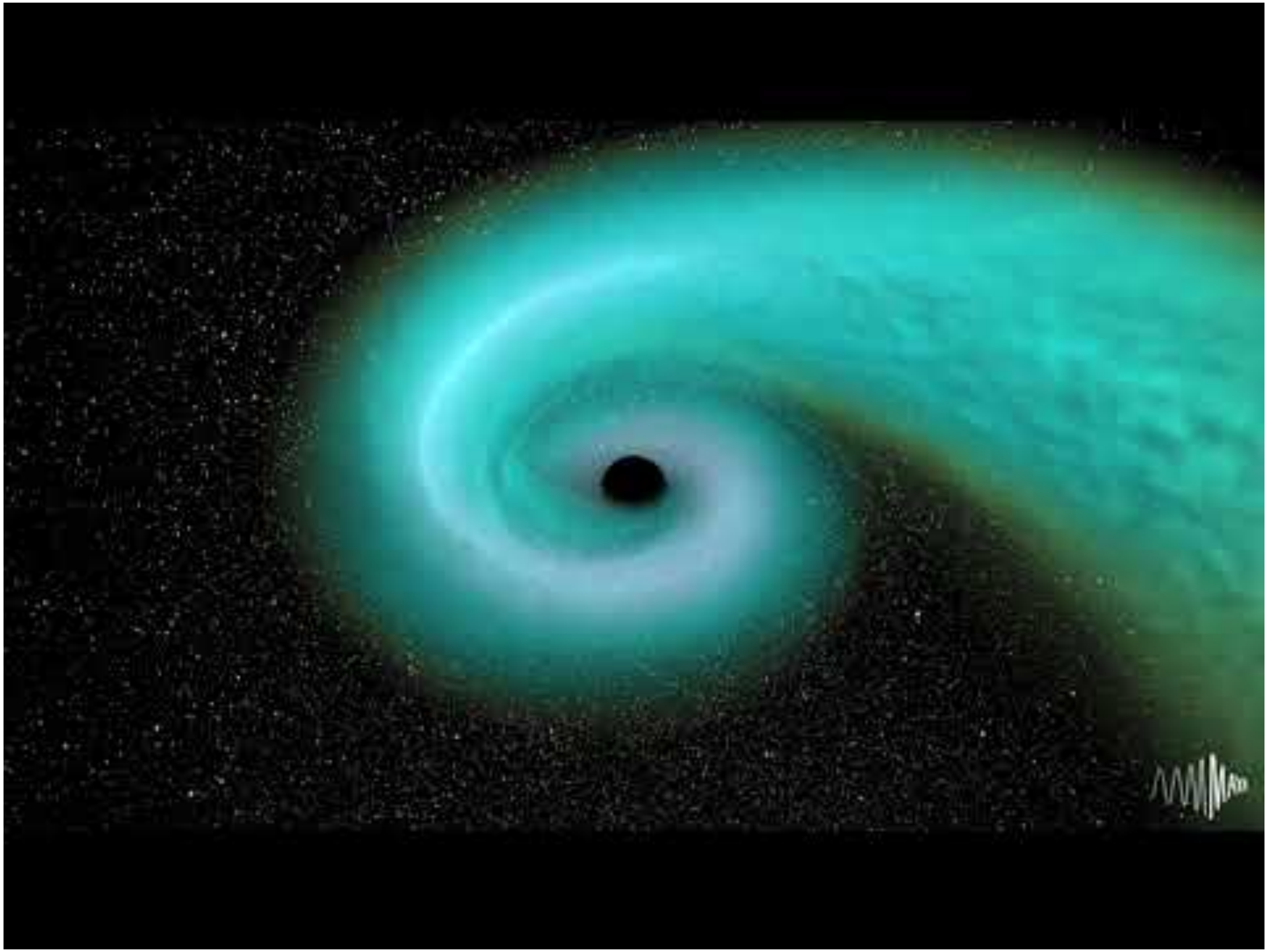
The quantities (α,β^i) can be set to just about anything: tremendous freedom to select them. **The vast majority of choices we can make are *terrible*.**

Over the (many) years in which people were beating their heads against the wall, they catalogued quite a few examples of gauge conditions that did not work well. Once all the ingredients were "mixed together," finding the "Holy Grail" turned to be something within reach of many groups.

Binary black holes now routine!



Frontier: including matter sources



Example: Neutron star in a binary with a black hole.

(Video: Ferguson, Khamesra, Jani; Maya collaboration.)

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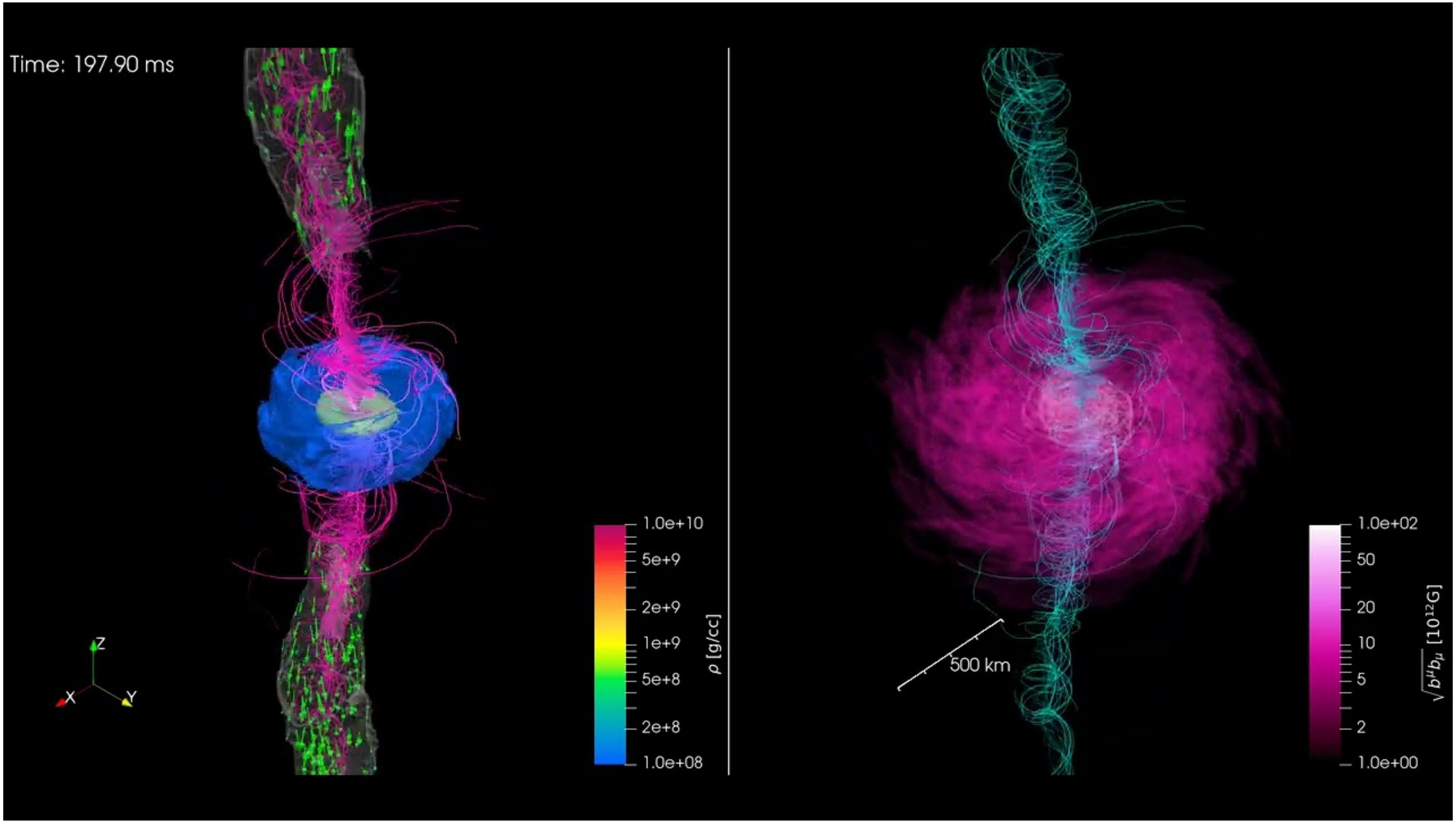

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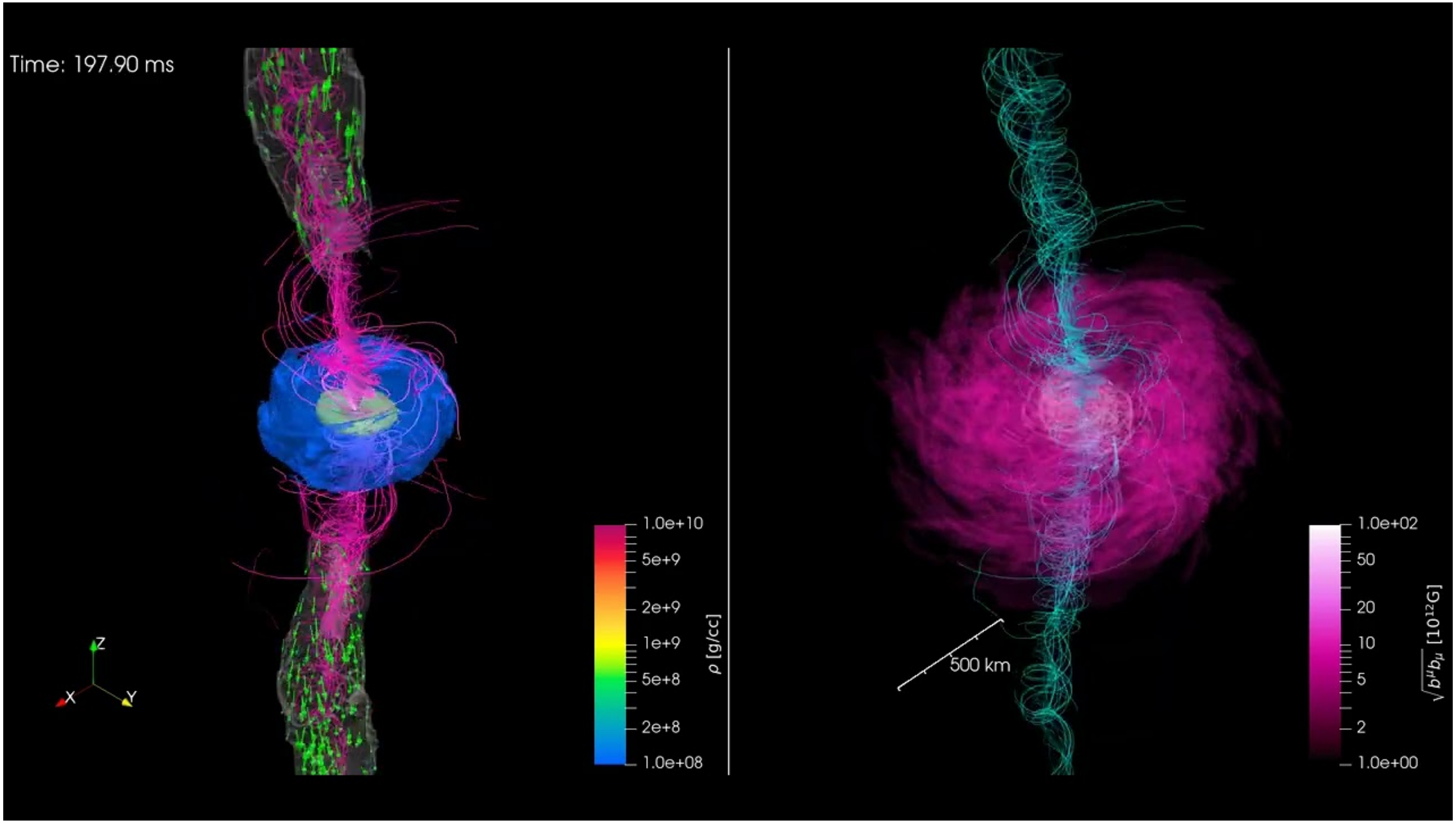

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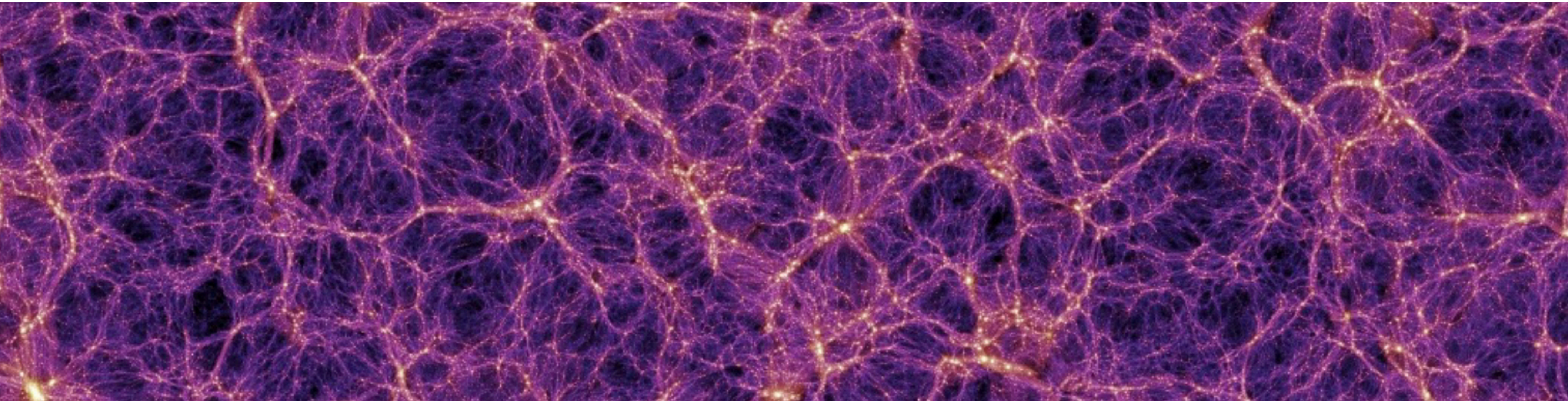
Young direction: Fully relativistic large-scale cosmology simulations



Long history of simulating growth of structure by essentially using Newtonian gravity on a cosmological background (ie, using Newton to compute how density contrast evolves).

Famous example (shown here) by Volker Springel and collaborators (including Mark Vogelsberger), used for a lot of cosmological inference over the years.

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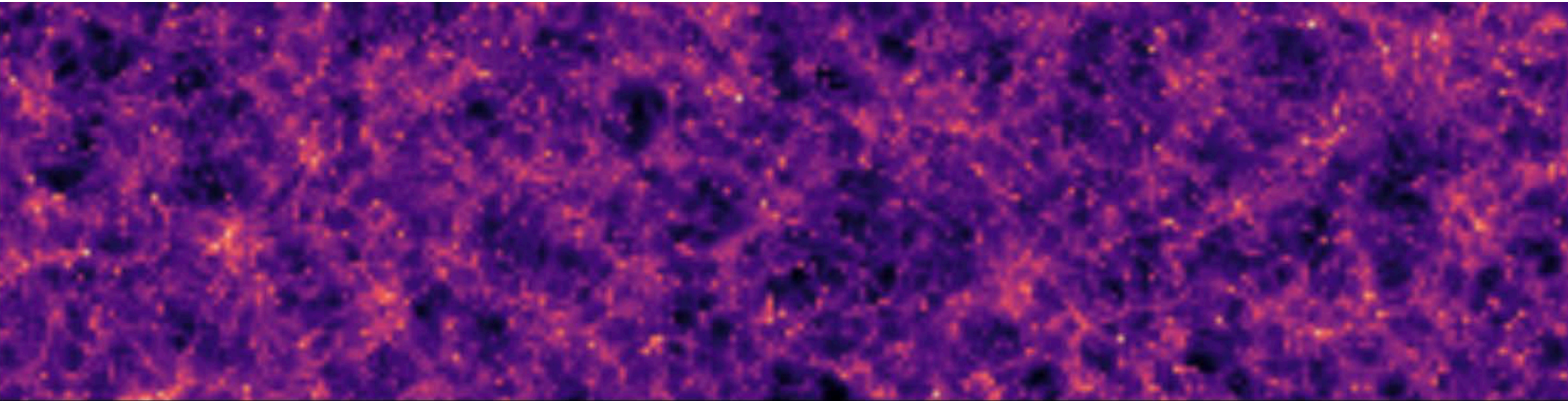

Now beginning to develop tools to do the whole thing self consistently in general relativity!

(This plot: Macpherson et al 2018)

May allow us to understand what scales are "large enough" for cosmological assumptions to be correct, and to assess consequences when they are not.