Surreal Numbers and Games

February 10, 2009

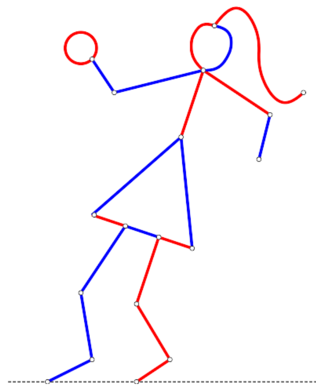
Introduction

Last week we began looking at doing arithmetic with impartial games using their Sprague-Grundy values. Today we'll look at an alternative way to represent games as numbers that we will extend to include *partisan* games as well.

Surreal numbers were introduced in Donald Knuth's (fiction) book *Surreal Numbers: How Two Ex-Students Turned on to Pure Mathematics and Found Total Happiness*, and the full theory was developed by John Conway after using the numbers to analyze endgames in GO. We'll start by using Conway's methods to represent games, and then show how these games/numbers form a new number system.

On Numbers and Games

To begin, we'll look at a partisan version of the impartial game of Green Hackenbush we saw last week. This game is called "Red-Blue Hackenbush." It is played similarly to Green Hackenbush, but now each line segment might be colored either red or blue. There are two players who for convenience in notation will be called L and R. On L's turn, he can only chop off blue branches, and R can only chop off red branches. As before, when a player removes a branch, all branches that are now disconncected from the "ground" also disappear. The player to chop off the last branch wins. The game below is an example:



Below we'll use this and other games to define games and surreal numbers.

What is a Game

A ***game*** (in combinatorial game theory) is defined as:

$$G = \{G^L | G^R\}$$

where G^L, G^R are sets of games themselves. This definition is recursive and can be confusing at first, so we'll look at many examples. So G is a set of sets of games. The base case is $\{\emptyset|\emptyset\}$ which will be called the ***endgame*** and occurs when neither player has any moves left.

The sets of games in G are the positions each player can move to. In an impartial game, since each player has the same options, both G^L and G^R will always be the same. In a partisan game, such as red-blue hackenbush, these options can be different.

First consider the red-blue hackenbush game in which both players have identical figures consisting of all branches of their own color. Each player has the same number of possible moves. The game will proceed as follows: the first player to move takes one branch of his color, the next player takes one branch of her color, and the game alternates back and forth until each player has only one branch left. The first player is forced to take his last branch, leaving the last branch on the page to the second player, who wins the game. We will call such a game, in which the second player to move wins, a ***zero position*** (equivalent to the P-position games with SG values of 0 from last week).

If the players had started with different numbers of branches, say 8 for L and 5 for R, then L would have a 3 move advantage, and we will say the value of this game G is 3. Similarly, if R has 3 more branches than L, the value of the game would be -3.

In general, we'll use the following ***outcome classes*** to describe games: (Note we use "always wins" to mean a player always has a winning strategy. Of course it is possible for them to make a mistake and lose.)

- $G = 0$ The second player to move always wins.
- $G < 0$ Player R always wins.
- $G > 0$ Player L always wins.

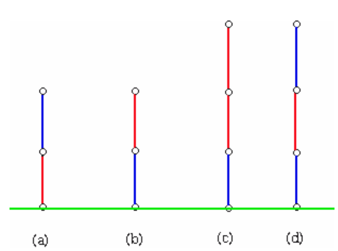
This seems to cover all possible values of G, but in many games we can imagine cases where the *first* player to move always wins. This type of game is neither less than, greater than, or equal to 0. Instead, we will call it *fuzzy* or *confused* with 0, denoted as $G||0$.

Consider the simple game consisting of a single blue branch. L has one move and R has none. This game is denoted as $G = \{0|\}$, since L can move to the 0 game and R has no moves. We will say this has a value of 1, since it is a one move advantage for L. Similarly, the game with a single red stalk has value $G = \{|0\}$ and is equal to -1. So far we have the following "numbers":

$$\{|\} = 0, \{0|\} = 1, \{|0\} = -1$$

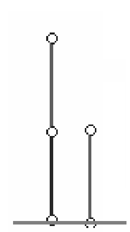
Similarly, any game of the form $\{n|\} = n + 1, \{|n\} = -n - 1$.

Can our games have fractional values? Look at the following red-blue hackenbush positions:



In figure b, if R moves first, he can only chop off the top branch, leaving 1 branch for L and a game with a value of 1 since L now has a one move advantage. If L moves, he can only chop off the bottom branch, leaving 0 branches, or the endgame. So the game is denoted as $G = \{\{\emptyset|\emptyset\}|1\}$. We can replace G^R with 0, since we defined the endgame to be so above, giving us: $G = \{0|1\}$

What is the value of G? We know it must be positive, since L clearly has the advantage in this game. But does L have a one-move advantage? If so, if we give R back an extra move then we would expect the value of the game to be 1 - 1 = 0, or a second player win. Let's see what happens:



Now if we were correct that the left stalk has a value of 1, then adding the red stalk with value -1 should make this a 0 game. If L goes first, he leaves two red branches with a value of -2. If R moves first he takes either branch, then L takes a branch, and there is still a branch left for R to take and win no matter what. Clearly this game has a negative value since R can win all the time. So what is the value of the original game? It turns out it is $\frac{1}{2}$, or a half move advantage for L. We can verify by adding two of these games together to see that they have a value of 1.

So now our list of numbers is:

$$\{|\} = 0, \{0|\} = 1, \{|0\} = -1, \{0|1\} = \frac{1}{2}, \{1|0\} = -\frac{1}{2}, \{n|\} = n{+}1, \{|-n\} = -n{-}1$$

What is a Surreal Number

We will call a ***form*** (game) $\{L|R\}$ numeric if there is no $x_L \in L$ and $x_R \in R$ such that $x_R \leq x_L$. So every number to the left of the | must be less than every number to the right of the —. (note NOT equal! we'll get to this case). Note that all surreal numbers can be games, but not all games can be surreal numbers. For instance, the games $\{0|0\}$ and $\{1|-3\}$ can be games, but are not numeric.

Similar to our definition of games, L and R are themselves sets of surreal numbers, so the definition is again recursive. Also note that different sets L and R may actually form the same number. We say that numeric forms are placed in ***equivalence classes***. So to be clear, the *forms* described above form equivalence classes, each of which is a (surreal) *number*.

The following are useful theorems about surreal numbers. Proofs can be found in the reading for the week:

1. Theorem 1: If x is a surreal number, then $x = x$.
2. Theorem 2: If $A = B$ and $C = D$, then $\{A|C\} = \{B|D\}$.
3. Theorem 3: A surreal number $X = \{X_L|X_R\}$ is greater than all members of its left set x_L and less than all members of its right set x_R.
4. Theorem 4: For the number $X = \{X_L|X_R\}$, we can remove any member of the left set x_L except the largest or any member of the right set x_R without changing the value of the number.

Comparing Real Numbers (and games)

The surreal numbers form a totally ordered field, and any two forms that are numeric can be compared to each other using the following rules:

- Given two numeric forms $x = \{X_L|X_R\}$ and $y = \{Y_L|Y_R\}$, we say $x \leq y$ iff there is no $x_L \in X_L$ such that $y \leq x_L$ and there is no $y_r \in Y_R$ such that $y_R \leq x$. The two numeric forms above are equal if $x \leq y$ and $y \leq x$.

Games that are not also numbers are tricky to order, so we often call them confused or fuzzy, as noted above. One example of such a game is $\{0|0\}$, the game in which either player can only move to the endgame, and so the first player to move wins. We denote this special game as $*$ since it comes up so often. This game is neither greater than, less than, or equal to 0. More about this soon.

Construction of Numbers (and Games)

Surreal numbers can be constructed from the base case $\{\emptyset|\emptyset\} = 0$ using the ***induction rule***. We start with the generation $S_0 = \{0\}$ in which 0 is constructed from just the emptyset (which from now on we will omit, so if there is no number in a set it is assumed to be the emptyset). Starting with this 0th generation, each new generation X_n consists of all of the (well-formed) surreal numbers generated from subsets of $\cup_{i<n} S_i$. We say that the numbers born in generation S_n all have the same birthday, or were born on day n. So 0 was born on day 0, and gave birth to all of the numbers that we know in the real number system.

In terms of games, the birthday of a number can be seen as the depth of the game in the game tree. So the endgame, or 0, born on day 0, is a game that's over. A game with value of 1 is 1 move into the game tree, etc.

S_1 is constructed from combinations of members of S_0, or the emptyset and 0. So possible members of S_1 are:

$$\{0|\}, \{0|0\}, \{|0\}$$

But the middle form is omitted since it is not numeric (0 is not less than 0), so we only have two new numbers. These are 1 and -1.

S_2 now consists of all combinations of 0, 1, -1, and the emptyset:

$$\{-1, 0|\}, \{-1, 0, 1|\}, \{-1|\}, \{|0, 1, -1\}, \{|0, 1\}, \{| -1\}, \{0|1\}, \{1|0\}, \{1| -1\}$$

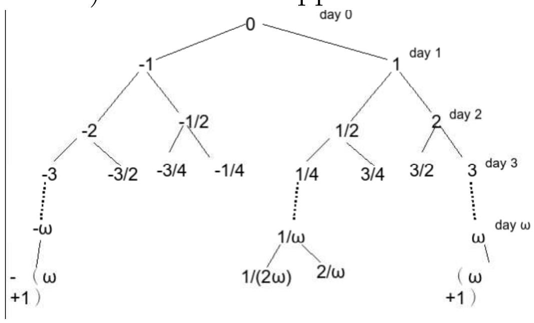
But we omit all of the non-well formed numbers and are left with four new numbers:

$$\{1|\}, \{| -1\}, \{0|1\}, \{-1|0\}$$

which will be called 2, -2, $\frac{1}{2}$, and $-\frac{1}{2}$, respectively.

So a pattern emerges. Every new generation S_n has at its extremal elements $-n$ and n as $\{|n-1\}, \{n-1|\}$, and all of the fractional numbers spaced equally in between all of the new elements and the previously existing numbers.

It seems that continuing in this manner can only give only all of the integers and all of the dyadic fractions (fractions with denominators as powers of two). But what happens if we extend our "tree" of numbers to infinity?



On day , we eventually come to the number $\omega = \{1, 2, 3, 4 \ldots |\}$ which is larger than all natural numbers. We also get $-\omega$, which is smaller than all natural numbers, as well as $\epsilon = \{0|1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8} \ldots\}$, which is the smallest positive number (the opposite for the largest negative number). So we have infinite and infinitesimal numbers in this generation, what else? Numbers in the generation S_ω may also belong to the familiar rational numbers. For instance, $\frac{1}{3} = \{\frac{1}{4}, \frac{1}{4} + \frac{1}{16}, \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \ldots | \frac{1}{2}, \frac{1}{2} - \frac{1}{8}, \ldots\}$. We also get transcendental numbers on day ω, for example $\pi = \{3, \frac{25}{8}, \frac{201}{64}, \ldots | 4, \frac{7}{2}, \frac{13}{4}, \frac{51}{16}, \ldots\}$. With a little playing around we can get to any number. We can even get beyond infinity to generations $\omega + n$.

Evaluating Surreals

So given a number $\{x|y\}$, how do we decide which surreal number it represents? $\{0|1\} = \frac{1}{2}$, so we might be tempted to just take the average of the largest number on the left and the smallest number on the right, but it turns out this fails. We see this with an example. Take the number $\{2\frac{1}{2}|4\frac{1}{2}\}$. The average of these numbers is $3\frac{1}{2}$, but we claim that the *number* it represents is 3. How do we test equality? We choose a form that we know is equal to 3, $\{2|\}$. Now we show that for $x = \{2\frac{1}{2}|4\frac{1}{2}\}$ and $y = \{2|\}$, both $x \leq y$ and $y \leq x$.

1. $x \leq y$.

    - $\nexists x_L \in X_L$ such that $y \leq x_L$. This is true, since the only x_L is $2\frac{1}{2}$, which is less than or equal to $y = 3$.

- $\nexists y_R \in Y_R$ such that $y_R \leq x$. This is vacuously true, since there is no member y_R since Y_R is empty!

2. $y \leq x$.

- There is no y_L, so always there is no y_L such that $x \leq y_L$.
- The only member of X_R is $4\frac{1}{2}$, which is not less than or equal to 3.

So we have proven that the value of x must be 3.

Below are some general forms you might find helpful:

1. $\{n|n+1\} = n + \frac{1}{2}$
2. $\frac{2p+1}{2^{n+1}} = \{\frac{p}{2^n}|\frac{p+1}{2^n}\}$, or in other words, each fraction with a denominator as a power of two has as its left and right options the two fractions nearest it on the left and right that have a smaller denominator which is again a power of two. So $\{\frac{1}{2}|\frac{3}{4}\} = \frac{5}{8}$.

In evaluating numbers and games we will use the ***Simplicity Rule***, which says that out of all the numbers between the largest member of the left set and the smallest number of the right set, the surreal number value of the form is the *simplest* number that fits, where we use simplest as meaning the number born earliest. This is just either the smallest integer between the two, or else the fraction between them having the highest power of two in the denominator.

(In class we'll do some hackenbush examples with the simplicity rule)

Arithmetic with Numbers (and Games)

We showed last week that it can be helpful to break games up into sums of smaller, easier to evaluate, games, and use the sum of the values of these games to describe the larger game. The same is true with surreal numbers and their representations of games.

The Negative of a Number

The negative of a number $x = \{X_L|X_R\}$ is $-x = \{-X_R| - X_L\}$. In terms of games, the negative of a game is just the game with the positions of L and R reversed. In a hackenbush game, for instance, just interchange all of the blue and red segments to get the negative of a game.

Addition

To add the numbers $x = \{X_L|X_R\}$ and $y = \{Y_L|Y_R\}$ we get:

$$x + y = \{X_L + y, x + Y_L|X_R + y, x + Y_R\}$$

where $X + y - \{x + y : x \in X\}, x + Y = \{x + y : y \in Y\}$. Below are a few examples:

- $0 + 0 = \{|\} + \{|\} = \{|\} = 0$
- $x + 0 = x + \{|\} = \{X_L + 0|X_R + 0\} = \{X_L|X_R\} = x$
- $\frac{1}{2} + \frac{1}{2} = \{0|1\} + \{0|1\} = \{0 + \frac{1}{2}, \frac{1}{2} + 0|1 + \frac{1}{2}, 1 + \frac{1}{2}\} = \{\frac{1}{2}|\frac{3}{2}\} = 1$

So in a sum of games $G = \{G_L|G_R\}$ and $H = \{H_L|H_R\}$, the sum of the games has as left options the value if L moves in G, $G_L + H$, (since H is unchanged) and $H_L + G$ if L moves in H. Similary, the right options are $G_R + H$ and $H_R + G$ depending on where R moves. This agrees with our notion of addition of surreal numbers. To subtract, just add the negative of a game.

Multiplication

Surreal numbers can also be multiplied, but we won't really use this definition in our game analysis. I include it here just to show that it can be done and to fit with our original statement that the surreal numbers form an ordered field.

To multiply $x = \{X_L|X_R\}$ and $y = \{Y_L|Y_R\}$:

$$xy = \{X_L y + xY_L - X_L Y_L, X_R y + xY_R - X_R Y_R | X_L y + xY_R - X_L Y_R, xY_L + XRy - X_R Y_L\}$$

Special Non-numeric Games

All of our arithmetic rules are defined for numbers, but we said before that a lot of games aren't well-formed, or in other words aren't actually numbers. Can we still do arithmetic with these games? The answer is yes!

$\{0|0\}$

Earlier we mentioned that we would call the game $\{0|0\}$ *. Consider the green hackenbush game with a single line segment left:
*picture here *
If left moves first, he takes the stalk and wins, and similarly right moves if he goes first. What happens if we add another disconnected green stalk?
*picture here *
This is the game of * + *, since it consists of the sum of two * games. First look at left's options. He can take either one of the two stalks, after which right takes the remaining one and wins. It is clear that the first player to go loses. But this is just our definition of a 0 game! So we get:

$$* + * = 0$$

What about adding * to a number x? To look at this position we will introduce a third type of hackenbush, red-green-blue hackenbush, in which there are red, green, and blue segments. The red and blue ones belong to right and left, respectively, but the green segments may be cut by either player. Consider the game consisting of a stalk of 2 blue segments and a stalk of one green segment. This is the game 2 + *. Left has three options: take the green stalk to leave the game with a value of 2, take the bottom blue stalk for a value of *, in which case right will win on the next move by taking the remaining green stalk, or take the top left stalk to leave the game of 1+*. The most positive of these options is clearly to take the green stalk to make a value of 2. Right has only one option: take the green stalk to create a game of value 2. So we end up with:

$$2 + * = \{2|2\}$$

Similarly, for any $x + *$, we get $\{x|x\}$. Often we omit the star from the expression $x + *$ and instead just write $x*$.

Arrows

To illustrate these next two values, we will introduce another game called Toads and Frogs. We will play the game for implicity on a strip of 5 squares starting with two toads on the left two squares and two frogs on the right two squares:

$$TT - FF$$

The toads can only move to the right one square each move and the frogs to the left. Toads and frogs can also jump over a toad or frog in an adjacent square to the next square over (as in Chinese Checkers). The game is over when a player can't make a move.

Consider the position $T - TFF$. Evaluating the possible T and F positions gives the game: $\{-TTFF|TFT - F\} = \{0|*\}$ since no one can move in the left option and the first player to move wins the right option. This value arises so often in describing games that it is given a special name, up, or $\uparrow$. Similarly, the opposite game $\{*|0\}$ is denoted as $\downarrow$. Since $\uparrow = -\downarrow$, we have $\uparrow + \downarrow = 0$.

Look at the starting position $G = TT - FF = \{T - TFF|TTF - F\} = \{\uparrow \mid \downarrow\}$. To evaluate this game we will look at the game $G - *$. Does L have a winning strategy from $\{\uparrow \mid \downarrow\} + \{0|0\}$? If he moves from * to 0, F will move in G to $\downarrow$, which favors R. If L moves to $\uparrow$, R will move from $\uparrow$ to * in G leaving *+*=0. So L does not have a winning strategy. We can show similarly that R also doesn't have a winning srategy going first. So that means $G - * = 0$. We can rewrite this as $G = *$ and get the following:

$$\{\uparrow \mid \downarrow\} = \{\uparrow |0\} = \{0| \downarrow\} = \{0|0\} = *$$

Simplifying Games

When looking at the options available to players in games, it is helpful when there are many options to be able to simplify things. We'll use the following processes to simplify:

1. Eliminate Dominated Options

2. Eliminate Reversible Options

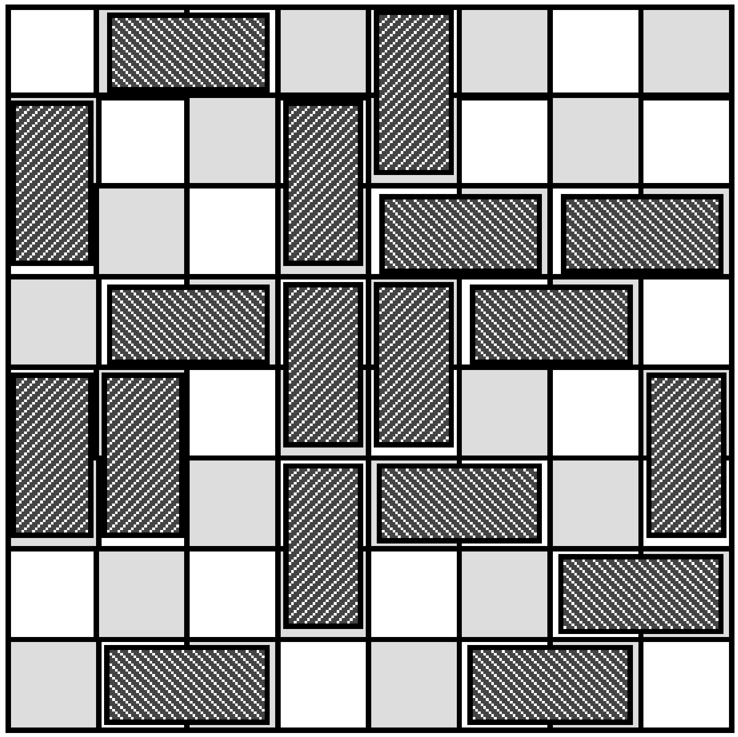
In a game $G = \{A, B, C, \ldots | D, E, F \ldots\}$ if $A \leq B$ or $D \leq E$ we say that A is dominated by B and E is dominated by D. We can eliminate these dominated options, since a logical player will only choose the best option available.

Now to reversible options. Consider the same above, where if right moves to D then left has some option D^L which is at least as good for left as G was to begin with, i.e. $D^L \geq G$. Then if right ever decides to move to G, then left can at least reverse the effect of this move by moving to D^L (and his position might even get better!). To get a better feel for this, consider the game of poker Nim. This game is similar to the Nim that we played last week. We play with three heaps of poker chips and can take any amount of chips from any single pile on a single turn. The only difference is that players are now also allowed to add any amount of chips to a single pile. (but not both add and subtract on the same turn). At first this game may seem more complicated than regular Nim, until we see that adding any amount of chips is a reversible move. Say player 1 adds x chips to a heap. Then player 2 can just take away x chips from the heap and leave player 1 back where he started. So the strategy for poker Nim is the exact same as for regular Nim, with the exception of removing any chips your opponent might add to the pile.

Games!

Domineering

The game of domineering is played on a checkerboard. Players alternate placing dominoes (the size of two checkerboard spaces) on the board. Left can only place his dominoes vertically, while Right can only place them horizontally. The first player to not be able to place a domino on the board loses. Try evaluating the values of the opening positions on the following partially played domineering board. The value of the game is the sum of the values of these empty positions:



Exercises

Try evaluating the Domineering and hackenbush positions on the worksheets. If you get stuck check them with the programs listed on the software page of the website.