

# **Random Unitary Processes and Incoherent Noise**

**N. Boulant, J. Emerson, T. H. Havel, D. G. Cory**

**S. Furuta\***

**October 2, 2003**

# Outline

1. Definition of random unitary process and decomposition (RUP/RUD)
2. Incoherent noise
3. Quantum process tomography: problems and ambiguities
4. Eigenvalue structure of superoperators
5. Summary and conclusions

# Quantum Operations

- Physical maps among density matrices: Restricted by physically motivated axioms:
  1.  $\text{Tr}(\Lambda(\rho)) = \text{Probability of process } \Lambda \text{ occurring on the state } \rho.$
  2.  $\Lambda$  is convex-linear on the set of density matrices.
  3.  $\Lambda$  is positive and  $(I \otimes \Lambda)$  is also positive on arbitrary extensions of  $\rho$ .
- *Definition:* Kraus operator sum representation:

$$\rho \mapsto \sum_{i=0}^{N^2-1} K_i \rho K_i^\dagger \quad (1)$$

$$N = \dim(\rho). \quad (2)$$

- *Definition:* A doubly stochastic quantum operation is unital and trace-preserving:

- Unital:  $\sum_i K_i K_i^\dagger = I.$

- TP:  $\sum_i K_i^\dagger K_i = I.$

# Random Unitary Process

- *Definition:* Random unitary process is a quantum operation of the form:

$$\rho \mapsto \sum_{i=0}^{N^2-1} p_i U_i \rho U_i^\dagger \quad (3)$$

$$\sum_i p_i = 1 \quad (4)$$

and in the continuous case:

$$\rho \mapsto \int d\alpha p(\alpha) U(\alpha) \rho U(\alpha)^\dagger. \quad (5)$$

$$\int d\alpha p(\alpha) = 1. \quad (6)$$

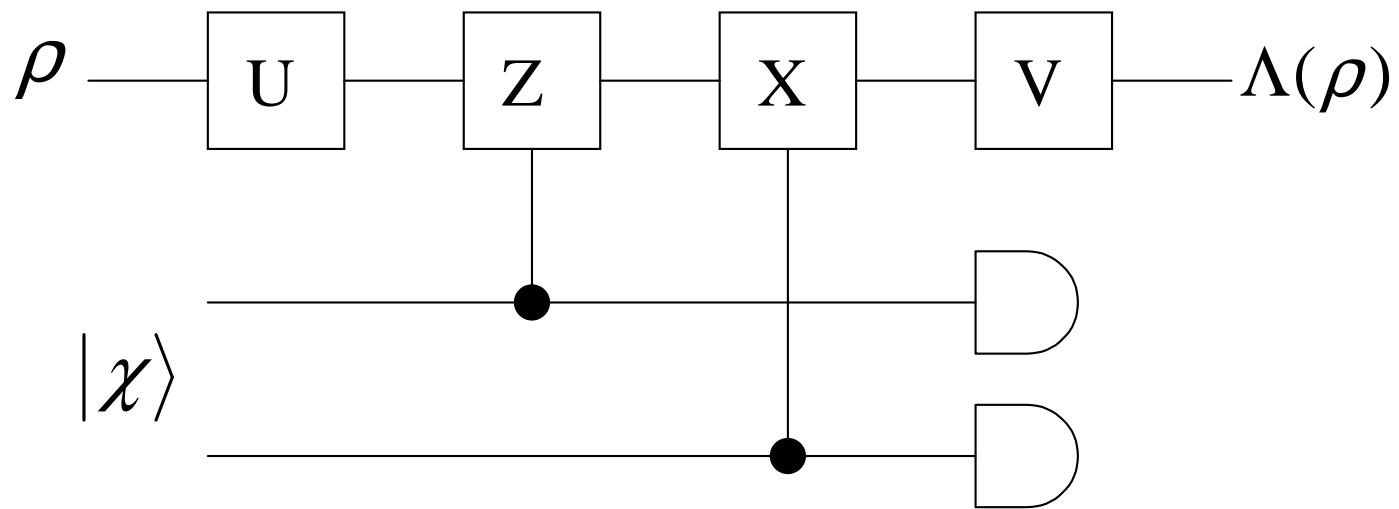
- *Definition:* Given a map  $\Lambda$ , a random unitary decomposition (RUD) of  $\Lambda$  is a set of pairs  $\{(p_i, U_i)\}$  that comprise an operator-sum representation of  $\Lambda$ .
- *Example:* One qubit<sup>a</sup>. All one qubit unital quantum operations have at least one RUD.
- There exist for more than 2 qubits non-RUD maps.<sup>b</sup>

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<sup>a</sup>Debbie Leung, PhD Thesis (2000); Ruskai, Werner, and Szarek (2000).

<sup>b</sup>Streater, Lin. Alg. Apps.

- Circuit:



- Set of all unital processes is a convex set, whose extreme points have been characterized by Choi<sup>a</sup>.
- Given a UP, we can test for extremality. If  $\Lambda$  is extremal and is not a unitary process, then  $\Lambda$  does not have a RUD.
- We have not seen necessary and sufficient conditions for the existence of an RUD, given  $\Lambda$ .

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<sup>a</sup>Choi, Lin. Alg. App. (1975).

# Incoherent Noise

$$H = H_S + H_E + V \quad (7)$$

$$V = V_S \otimes \chi(\hat{z}) \quad (8)$$

Assume  $I \otimes \rho_E(0)$  is an eigenstate of  $V$ .

$$\rho_S(t) = \text{Tr}_E \left( e^{-i(H_S + H_E + V)t} \rho_S(0) \rho_E(0) e^{i(H_S + H_E + V)t} \right) \quad (9)$$

$$\tilde{\rho}_S(t) = \text{Tr}_E \left( e^{-i(H_S + \tilde{V})t} \rho_S(0) \rho_E(0) e^{i(H_S + \tilde{V})t} \right) \quad (10)$$

Long correlation time of the environment:  $H_E t \ll 1$ ,  $\tilde{V} \approx V$ .

$$\rho_S(t) = e^{-i(H_S + V_S \chi(z))t} \rho_S(0) e^{i(H_S + V_S \chi(z))t} \quad (11)$$

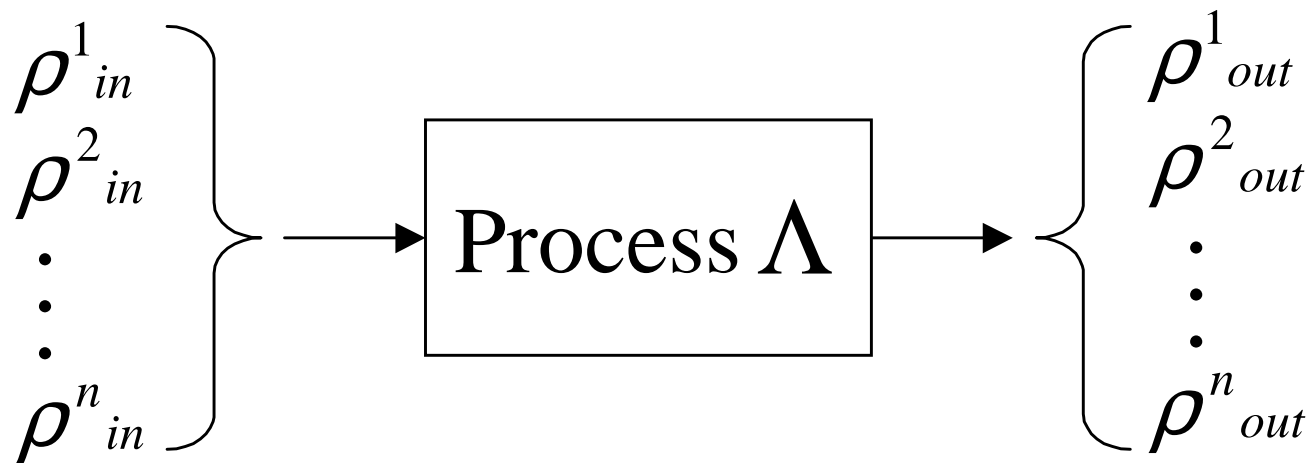
$$\rho_E(0) = \int dz p(z) |z\rangle \langle z| \quad (12)$$

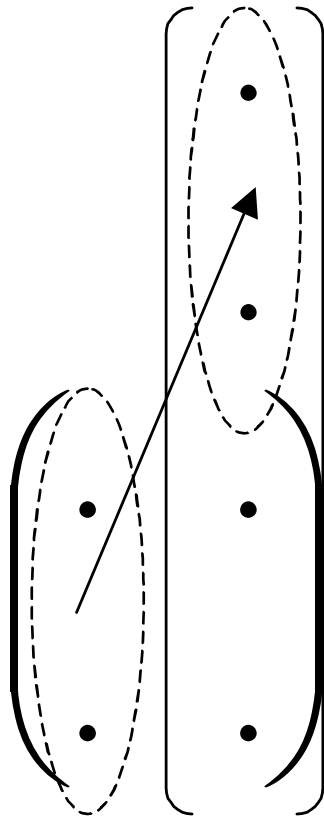
$$\rho_S(t) = \int dz p(z) U(z, t) \rho_S(0) U^\dagger(z, t) \quad (13)$$

(RUP)

1. Initial state of the environment is in an eigenstate of the coupling Hamiltonian.
2. Environment evolution during the time of the experiment is small.

# Incoherent Noise with Quantum Process Tomography





- Columnization of  $\rho$ : Stack columns from left to right.
- Allows  $\Lambda$  to be represented by a matrix acting on column vectors.

$$S = \sum_i \bar{K}_i \otimes K_i.$$

- QPT amounts to computing:

$$S = B_{\text{out}} B_{\text{in}}^{-1} \quad (14)$$

where  $B$  is a matrix whose columns are the columnized- $\rho$  vectors.

- This procedure works when system and environment are in a separable state.
- If there is initial entanglement between system and environment, the Kraus operator sum representation fails to tell the whole story.
- A general qubit-qubit state can be parametrized by

$$\rho_{SE} = \frac{1}{4} (I_{SE} + \alpha_i \sigma_S^i \otimes I + \beta_j I \otimes \sigma_E^j + (\gamma_{ij} + \alpha_i \beta_j) \sigma_S^i \otimes \sigma_E^j) \quad (15)$$

$$\rho_S = \frac{1}{2} (I + \alpha_i \sigma_i) \quad (16)$$

$$\rho_E = \frac{1}{2} (I + \beta_i \sigma_i) \quad (17)$$

$$\rho'_S = \sum_{\mu\nu} M_{\mu\nu} \rho_S M_{\mu\nu}^\dagger \quad (18)$$

$$+ \sum_{\mu} \langle \mu | U_{SE} \gamma_{ij} \sigma_i \otimes \sigma_j U_{SE}^\dagger | \mu \rangle \quad (19)$$

The second term involves only the correlation  $\gamma_{ij}$ .<sup>a</sup>

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<sup>a</sup>Stelmachovic and Buzek (2001)

## Two Examples

1.  $U_{SE} = \text{SAWP}$ : Does not change the initial correlations  $\longrightarrow$  no contribution from the second term even when  $\gamma_{ij} \neq 0$ .

2.  $U_{SE} = \text{CNOT}$ :

$$\rho_{SE}^{(1)} = |\alpha|^2 |00\rangle\langle 00| + |\beta|^2 |11\rangle\langle 11| \quad (20)$$

$$\rho_{SE}^{(2)} = (\alpha |00\rangle + \beta |11\rangle)(\text{h.c.}) \quad (21)$$

3. Both  $\rho_{SE}^{(1)}$  and  $\rho_{SE}^{(2)}$  yield the same reduced states, but  $\rho_{SE}^{(2)}$  contains correlations.

4. In contrast, under CNOT, the system state becomes

$$\rho_S^{(1)'} = \frac{1}{2}(I + \sigma_3) \quad (22)$$

$$\rho_S^{(2)'} = \frac{1}{2}(I + (|\alpha|^2 - |\beta|^2)\sigma_3) \quad (23)$$

## Ambiguities for QPT

- Noise can introduce initial correlations between  $S$  and  $E$ . If the correlation time of  $E$  is large compared to duration of experiments the correlations persist and may render the superoperator non-CP.
- Lose linearity in  $\rho$  - different initial states have different correlations.

For example: We take a set of 4 initial density matrices  $\rho_{SE}$  such that  $\rho_E$  is the same in each case and  $\rho_S$  span the state space of  $S$  (as required by QPT)

$$\rho_{AB}^1 = (I_{AB} + \beta I \otimes \sigma_z)/4$$

$$\Rightarrow \rho_A = I/2$$

$$\rho_{AB}^2 = (I_{AB} + \alpha \sigma_x \otimes I + \beta I \otimes \sigma_z + \gamma \sigma_x \otimes \sigma_z)/4$$

$$\Rightarrow \rho_A = (I + \alpha \sigma_x)/2$$

$$\rho_{AB}^3 = (I_{AB} + \alpha \sigma_y \otimes I + \beta I \otimes \sigma_z + \gamma \sigma_y \otimes \sigma_z)/4$$

$$\Rightarrow \rho_A = (I + \alpha \sigma_y)/2$$

$$\rho_{AB}^4 = (I_{AB} + \alpha \sigma_z \otimes I + \beta I \otimes \sigma_z + \gamma \sigma_z \otimes \sigma_z)/4$$

$$\Rightarrow \rho_A = (I + \alpha \sigma_z)/2$$

where in each case  $\rho_B = (I + \beta \sigma_z)/2$ .

Given  $U_{SE} = e^{-i\frac{\pi}{4}\sigma_z \otimes \sigma_z}$ , we find the corresponding 4 outputs

$$\begin{aligned}\rho_{out}^1 &= I/2 \\ \rho_{out}^2 &= (I + \gamma\sigma_y)/2 \\ \rho_{out}^3 &= (I - \gamma\sigma_x)/2 \\ \rho_{out}^4 &= (I + \alpha\sigma_z)/2.\end{aligned}$$

We write density matrices as vectors in the Zeeman basis and formulate a matrix equation.  $S$  is obtained by inverting the “input” matrix.

Find Choi matrix of  $S$  by

$$Choi(S) = \sum_{ij} (E_{ij} \otimes I) S (I \otimes E_{ij}). \quad (24)$$

A negative eigenvalue of  $\text{choi}(S)$  indicates that  $S$  is non-CP.

A CP-Filtering method <sup>a</sup> has been proposed: Remove the negative eigenvalues and corresponding eigenvectors from the Choi matrix and renormalize.

1. Ex1: (i)  $\alpha = \beta = 0.5$  and  $\gamma = 0.6$ . Negative eigenvalues  $\Rightarrow$  non-CP. After CPF, one unitary Kraus operator. (ii)  $\alpha = \beta = 0.5$  and no correlations. CP with two Kraus operators.
2. Ex2: (i)  $\alpha = \beta = \gamma = 0.5$ . CP with one unitary Kraus operator. (ii)  $\alpha = \beta = 0.5$  and no correlations. CP with two Kraus operators.

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<sup>a</sup>T. F. Havel, J. Math. Phys. (2003)

To Summarize:

	CPF	Correlations	CP	Num. Kraus Ops
Ex 1	X	✓	X	-
	✓	✓	✓	1
	X	X	✓	2
Ex 2	X	✓	✓	1
	X	X	✓	2

# Eigenvalues of the superoperator

General results:

- A CP superoperator  $S$  has a spectrum of eigenvalues whose structure on the Argand diagram is invariant under reflections in the real axis.
- In the Zeeman basis,  $S$  is a unitary matrix, CP and trace-preserving iif  $S = \bar{U} \otimes U$ .

In general, the unitary part of a CP superoperator (obtained e.g. by polar decomposition) will not be a valid CP unitary operation.

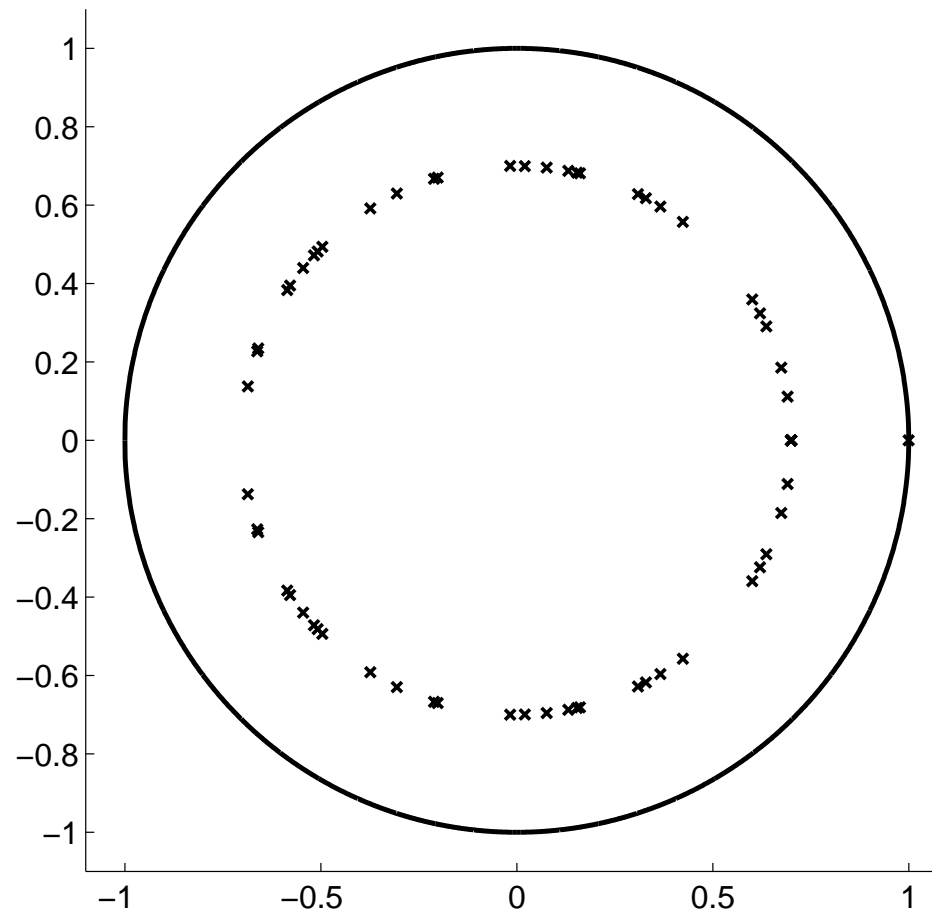
Example: Depolarizing channel.

$$\epsilon(\rho) = \frac{pI}{2} + (1-p)\rho, \quad (25)$$

$$= \left(1 - \frac{3p}{4}\right)\rho + \frac{p}{4}(\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z). \quad (26)$$

The superoperator has one eigenvalue at 1 and the rest at  $1 - p$ .

If we now combine the depolarizing channel with some other unital superoperator  $S_{rem}$ , we obtain  $S = S_{dep}S_{rem}$ . In this case, the eigenvalues of  $S_{rem}$  are uniformly attenuated by  $1 - p$  (except those at 1).



$S_{rem}$  is unitary. Eigenvalue structure simple and easy to identify.

# Perturbation theory analysis of the eigenvalue spectra

- Boundary between incoherent and decoherent processes is practical: it depends on the control resources available, i.e. whether a correlation time can be considered short or long.
- Both can be represented by CP maps, but incoherent processes are in principle refocusable.
- Incoherent process is a RUP.
- We can characterize the RUP by an inhomogeneity profile. This aids the experimenter to assess whether the noise can be counteracted given the control resources available.
- Extract via perturbation analysis.

Although the method applies more generally, we keep in mind the example of RF inhomogeneity in NMR QIP.

Different molecules in the sample see slightly different strength of RF pulses due to the spatial extent of the sample. Therefore different unitaries.

For a perfect pulse, one unitary.  $S$  is unitary and all eigenvalues lie on the unit circle.

$$U_0|\phi_j\rangle = e^{-i\phi_j t}|\phi_j\rangle, \quad (27)$$

$$H(\Delta\omega) = H_0 + K(\Delta\omega) \quad (28)$$

New eigenvalues of  $H(\Delta\omega)$  are

$$\tilde{\phi}_j(\Delta\omega) = \phi_j + \langle\phi_j|K(\Delta\omega)|\phi_j\rangle. \quad (29)$$

We now imagine the scenario where  $K(\Delta\omega)t = \Delta\omega K$ .

Setting  $K_{jm} = (\langle\phi_j|K|\phi_j\rangle - \langle\phi_m|K|\phi_m\rangle)$ ,

the perturbed eigenvalues become

$$\lambda_{jm} = e^{-i(\phi_j - \phi_m)t} \int p(\Delta\omega) e^{-iK_{jm}\Delta\omega} d\Delta\omega. \quad (30)$$

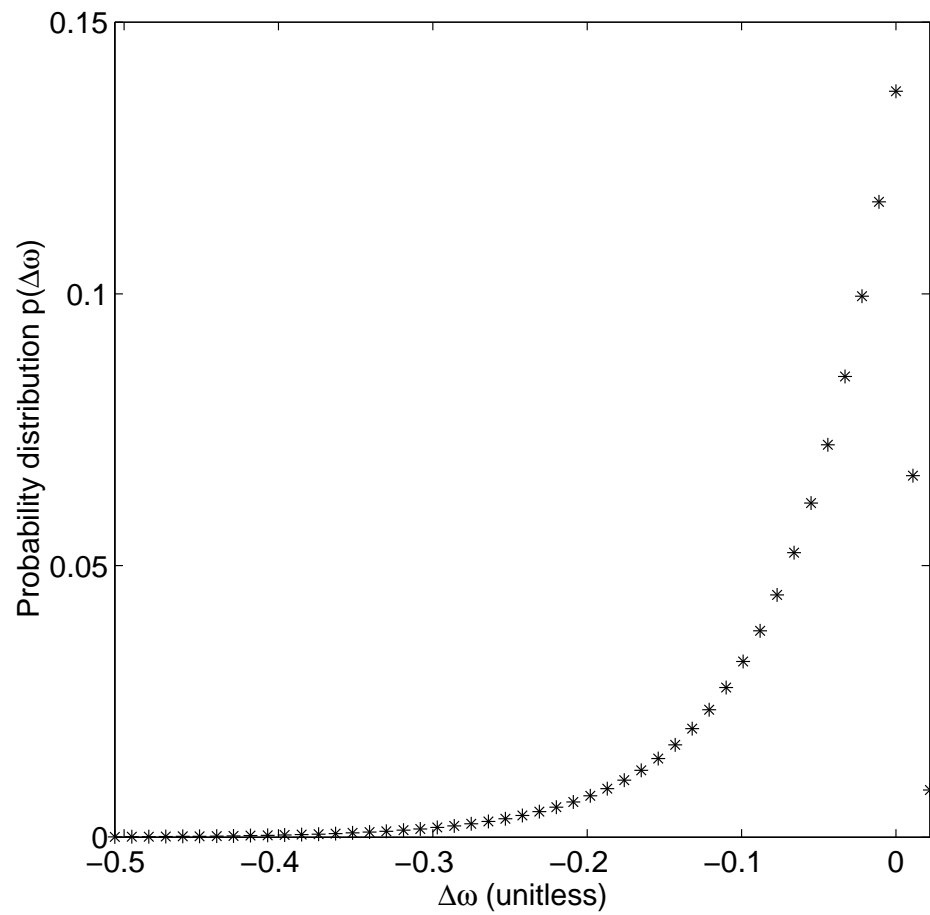
We see in this case that to first order the eigenvalue  $\lambda_{jm}$  is just the unperturbed eigenvalue  $e^{-i(\phi_j - \phi_m)t}$  times the Fourier transform of the RF distribution profile evaluated at  $K_{jm}$ .

## Recovery of the profile

Aim: recover  $p(\Delta\omega)$  from the eigenvalues of the measured superoperator.

- $\lambda_{jm}$  is the data.
- A model  $K$  is needed for the perturbation.
- $\phi_j$  and eigenvectors known through  $H_0$ .
- By Fourier transforming the data multiplied by  $e^{i(\phi_j - \phi_m)t}$  we obtain  $p(\Delta\omega)$ .

We demonstrate by simulating a 3-qubit system. This provides 64 eigenvalues.





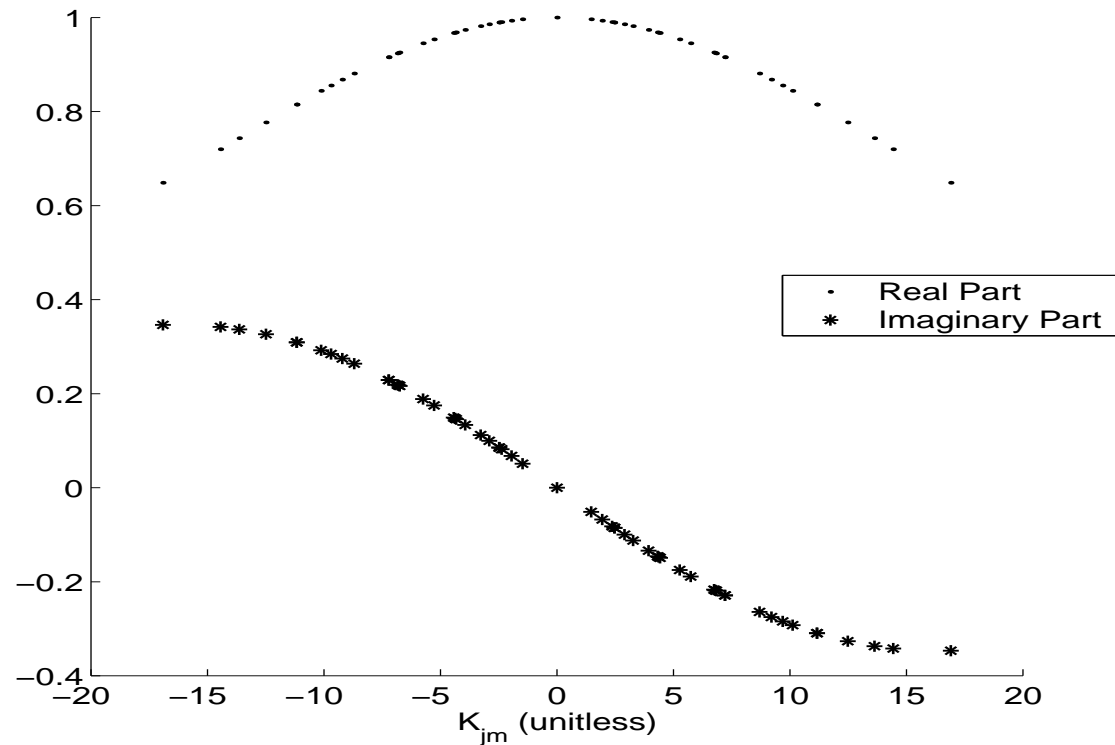


Figure 1: Plot of  $f = \int p(\Delta\omega)e^{-i\Delta\omega K_{jm}}$  with respect to  $K_{jm}$  (Real and Imaginary parts). The function is conjugate symmetric as expected, so that its inverse Fourier transform, which should be a probability distribution, is real.

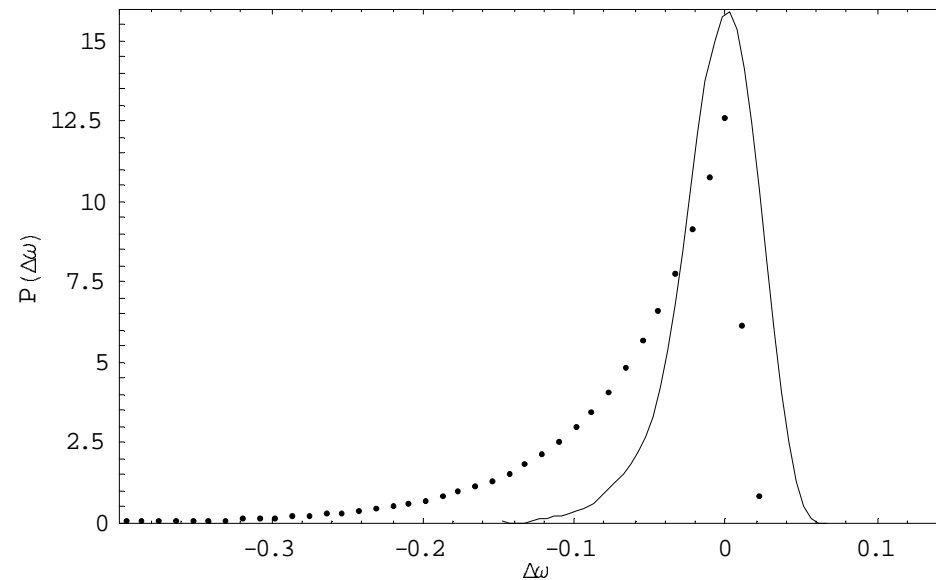


Figure 2: Inverse Fourier transform of the function plotted in previous figure shown together with points from the input inhomogeneity. The width of the profile in addition to its asymmetry are recovered.

# Summary and conclusions

- Defined incoherent processes.
- In addition to undesired non-unitary features, they also raise ambiguities in QPT.
- New method to extract a model for the incoherent process.
- Can gain physical insight into the underlying processes by analysing the eigenvalue spectra of the superoperators. E.g. Depolarizing channel, RF inhomogeneity.
- QPT on 3-qubits is experimentally feasible. The methods presented are testable.
- Better knowledge of incoherent processes allow experimenter to counteract noise using control pulses.