Lorentz Covariant Formulations of Multi-Qbit Dynamics via Geometric Algebra:

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Qbit's & Bloch Vectors

- A **Qbit** (quantum bit) is a quantum system with a Complete System of Commuting Observables of just a single 2D observable: its orientation along an axis, or **polarization**.
- The **Pauli matrix** basis for a Qbit's Algebra of Observables is

$$\boldsymbol{\sigma}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ \ \boldsymbol{\sigma}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ \ \boldsymbol{\sigma}_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix},\ \ \boldsymbol{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Its Bloch vector is a real linear combo thereof:

$$\boldsymbol{\rho} = \tfrac{1}{2} + x\boldsymbol{\sigma}_1 + y\boldsymbol{\sigma}_2 + z\boldsymbol{\sigma}_3 = \tfrac{1}{2}\begin{bmatrix} 1+z & x-iy \\ x+iy & 1-z \end{bmatrix}$$

Measurements & Bases

- Let $Z_\delta \equiv (1 + (-1)^\delta \boldsymbol{\sigma}_3)/2 \equiv |\delta\rangle\langle\delta|$ be the idempotents for projection along $\pm\boldsymbol{\sigma}_3$
- A measurement of a density operator $\boldsymbol{\rho}$ along the z-axis, i.e. of $\boldsymbol{\sigma}_3$, yields the density operator

$$Z_0 \boldsymbol{\rho} Z_0 + Z_1 \boldsymbol{\rho} Z_1 = \frac{1}{2}\left(1 + (\boldsymbol{\sigma}_3 \boldsymbol{\cdot} \boldsymbol{r})\, \boldsymbol{\sigma}_3\right)$$

- The two terms on the l.h.s. are the two possible outcomes: up ($|0\rangle$) with probability $\|Z_0 \boldsymbol{\rho} Z_0\|$, and down ($|1\rangle$) with probability $\|Z_1 \boldsymbol{\rho} Z_1\|$
- Note $|0\rangle = [1,0]$ & $|1\rangle = [0,1]$ are orthogonal eigenvectors of $\boldsymbol{\sigma}_3$ in Hilbert space with eigenvalues ±1

Enter Geometric Algebra

- The Pauli algebra of a qubit may be viewed as a matrix representation of the *geometric algebra* of 3D Euclidean vector space $\mathcal{G}(3)$, in which case:
  - the Bloch vector $\boldsymbol{r}$ is a real 3D vector;
  - $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$ are an orthonormal basis;
  - replace "i" by pseudo-scalar $I \equiv \boldsymbol{\sigma}_1\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3$;
  - $I[\boldsymbol{\rho}, H]$ is the cross product of $\boldsymbol{r}$ & $\boldsymbol{H}$
- This gives us with a classical (i.e. deterministic) geometric interpretation of the mean values, e.g. the gyroscopic precession of spinning top

Spinors & Rotations

- For time-independent $\boldsymbol{H}$, integration yields:

$$\exp(t\,ad_{-IH})\,\boldsymbol{\rho} \;=\; Ad_{\exp(-IHt)}\,\boldsymbol{\rho} \;\equiv\; e^{-IHt}\,\boldsymbol{\rho}\,e^{IHt}$$

which is the usual GA (or quaternion) formula for rotation about the axis $\boldsymbol{H}$ at angular speed $2\|\boldsymbol{H}\|$

- In particular, any pure state can be written as

$$\boldsymbol{\rho} = \boldsymbol{R}(1 + \boldsymbol{\sigma}_3)\boldsymbol{R}^\dagger / 2 \qquad (\boldsymbol{R} = \exp(\boldsymbol{IX}) \in \mathcal{G}^+(3))$$

i.e. as rotation of a reference $\boldsymbol{\sigma}_3$ to Bloch vector

- Following Hestenes, if the reference is clear we may identify the state with the "spinor" $\boldsymbol{R}$ itself

The Space-Time Algebra

- As shown by Hestenes, the Dirac algebra's even subalgebra $\mathcal{G}^{+}(1,3)$ is isomorphic the Pauli algebra
- A mapping between them is obtained by taking a time-like vector $\boldsymbol{\gamma}_0$ as the proper velocity of the observer, and the **relative spatial bivectors** as:

$$\boldsymbol{\sigma}_1 \equiv \boldsymbol{\gamma}_1\boldsymbol{\gamma}_0,\ \ \boldsymbol{\sigma}_2 \equiv \boldsymbol{\gamma}_2\boldsymbol{\gamma}_0,\ \ \boldsymbol{\sigma}_3 \equiv \boldsymbol{\gamma}_3\boldsymbol{\gamma}_0.$$

- Thus our 3D Euclidean interpretation of the mean values extends naturally to a relativistic one:

$$I\boldsymbol{\sigma}_1 = \boldsymbol{\gamma}_3\boldsymbol{\gamma}_2,\ \ I\boldsymbol{\sigma}_2 = \boldsymbol{\gamma}_1\boldsymbol{\gamma}_3,\ \ I\boldsymbol{\sigma}_3 = \boldsymbol{\gamma}_2\boldsymbol{\gamma}_1,\ \ I = \boldsymbol{\gamma}_0\boldsymbol{\gamma}_1\boldsymbol{\gamma}_2\boldsymbol{\gamma}_3.$$

The Multiparticle STA

- The MSTA is generated by a **direct sum** of N copies of space-time, $\mathcal{G}(N,3N)$, one per qubit.
- Bivectors from different factors **commute**, e.g.

$$\boldsymbol{\sigma}_x^a \boldsymbol{\sigma}_y^b = -\boldsymbol{\gamma}_x^a \left(\boldsymbol{\gamma}_y^b \boldsymbol{\gamma}_0^a\right) \boldsymbol{\gamma}_0^b = -\left(\boldsymbol{\gamma}_y^b \boldsymbol{\gamma}_x^a\right)\left(\boldsymbol{\gamma}_0^b \boldsymbol{\gamma}_0^a\right)$$

$$= \boldsymbol{\gamma}_y^b \left(\boldsymbol{\gamma}_0^b \boldsymbol{\gamma}_x^a\right) \boldsymbol{\gamma}_0^a = \boldsymbol{\sigma}_x^b \boldsymbol{\sigma}_y^a$$

- So the algebra generated by the $\boldsymbol{\sigma}$'s is a **tensor product** of the qubits' even subalgebras, $\mathcal{G}^{\otimes N}(3)$
- And we have **derived** the well-known fact that the state space of multiparticle systems is the tensor product of their individual state spaces

Multiparticle Spinors

- A (2^{N+1})D left-ideal that transforms one-sidedly under rotations of the qubits is generated by an idempotent called the **quantum correlator**:

$$\boldsymbol{E} \equiv \tfrac{1}{2}\left(1 - \boldsymbol{I}^1\boldsymbol{\sigma}_3^1\boldsymbol{I}^2\boldsymbol{\sigma}_3^2\right)\tfrac{1}{2}\left(1 - \boldsymbol{I}^1\boldsymbol{\sigma}_3^1\boldsymbol{I}^3\boldsymbol{\sigma}_3^3\right)\cdots\tfrac{1}{2}\left(1 - \boldsymbol{I}^1\boldsymbol{\sigma}_3^1\boldsymbol{I}^N\boldsymbol{\sigma}_3^N\right)$$

- For example, the two-particle basis states are

$$|00\rangle \leftrightarrow \boldsymbol{E},\ \ |01\rangle \leftrightarrow -\boldsymbol{I}^2\boldsymbol{\sigma}_2^2\boldsymbol{E},\ \ |10\rangle \leftrightarrow -\boldsymbol{I}^1\boldsymbol{\sigma}_2^1\boldsymbol{E},\ \ |11\rangle \leftrightarrow \boldsymbol{I}^1\boldsymbol{\sigma}_2^1\boldsymbol{I}^2\boldsymbol{\sigma}_2^2\boldsymbol{E}$$

- A general two-particle spinor can be written as:

$$\boldsymbol{E}\,a - \boldsymbol{I}^2\boldsymbol{\sigma}_2^2\boldsymbol{E}\,b - \boldsymbol{I}^1\boldsymbol{\sigma}_2^1\boldsymbol{E}\,c + \boldsymbol{I}^1\boldsymbol{\sigma}_2^1\boldsymbol{I}^2\boldsymbol{\sigma}_2^2\boldsymbol{E}\,d$$

with $a \equiv a_{\mathrm{r}} + a_{\mathrm{i}}\,\boldsymbol{J}$ etc. & $\boldsymbol{J} \equiv \boldsymbol{E}(\boldsymbol{I}^1\boldsymbol{\sigma}_3^1) = \cdots = \boldsymbol{E}(\boldsymbol{I}^N\boldsymbol{\sigma}_3^N)$

A Pseudoscalar Correlator

- Strangely enough, dim($\mathcal{G}^{\otimes N}(3)$) = 2^{3N} while the complex matrix algebras have real dim = 2^{2N+1}; this is because $\mathcal{G}^{\otimes N}(3)$ contains N algebraically indistinguishable pseudoscalars $\boldsymbol{I}^k$ ($k = 1, \ldots, N$)
- The dimension of $\mathcal{G}^{\otimes N}(3)$ may be painlessly reduced by working in an ideal defined by an idempotent from the center of the algebra:

$$\boldsymbol{C} \equiv \tfrac{1}{2}\left(1 - \boldsymbol{I}^1\boldsymbol{I}^2\right)\tfrac{1}{2}\left(1 - \boldsymbol{I}^1\boldsymbol{I}^3\right)\cdots\tfrac{1}{2}\left(1 - \boldsymbol{I}^1\boldsymbol{I}^N\right)$$

- Since $\boldsymbol{I}^k\boldsymbol{I}^\ell\, C = -C$, $\mathcal{G}^{\otimes N}(3)C$ is isomorphic to $\mathbb{C}(2^N)$, but even stranger: dim($\mathcal{G}^{+\otimes N}(3)C$) = dim($\mathcal{G}^{+\otimes N}(3)$)

Quaternionic Tensors

- Instead of looking upon $\mathcal{G}^{\otimes N}(3)C$ as a correlated tensor product of real algebras, we prefer to see it as a tensor product of complexified quaternion algebras with imaginary unit $\boldsymbol{I}^k C = \boldsymbol{I}^\ell C \equiv \boldsymbol{\iota}$
- Then the Pauli-even subalgebra $\mathcal{G}^{+\otimes N}(3) \approx \mathcal{G}^{+\otimes N}(3)$ is the subalgebra generated by tensor products of the real quaternions (& $\boldsymbol{\sigma}_\mu = -\boldsymbol{\iota\iota\sigma}_\mu$ is complex)
- The quaternions are generators of rotations, but for 2-particle <u>interactions</u> we must exponentiate $-\boldsymbol{\iota\sigma}_\mu^k \boldsymbol{\sigma}_\kappa^l$ for a compact group action (since $\boldsymbol{\iota\sigma}_\mu^k \boldsymbol{\iota\sigma}_\kappa^l$ squares to 1 not –1) and <u>project back into</u> $\mathcal{G}^{+\otimes N}(3)$

A Parity-Even Density Op.

- For one Qbit, we can map the usual self-adjoint (aka Pauli-reverse even) density operator onto the **parity-even** $\mathcal{G}^{\otimes N}(3)$ (w/wo C) as follows:

$$\tfrac{1}{2}(1+\boldsymbol{r}) \equiv \boldsymbol{\rho} \Leftrightarrow \boldsymbol{\varrho} \equiv \tfrac{1}{2}(1+\boldsymbol{\iota r})$$

- The mappings between the two can be written as

$$\boldsymbol{\rho} = \tfrac{1}{2}(\boldsymbol{\varrho}+\tilde{\boldsymbol{\varrho}}) - \tfrac{\boldsymbol{\iota}}{2}(\boldsymbol{\varrho}-\tilde{\boldsymbol{\varrho}}) \quad \Leftrightarrow \quad \boldsymbol{\varrho} = \tfrac{1}{2}(\boldsymbol{\rho}+\hat{\boldsymbol{\rho}}) + \tfrac{\boldsymbol{\iota}}{2}(\boldsymbol{\rho}-\hat{\boldsymbol{\rho}})$$

  where the tilde is reversion & hat the parity op.

- For multiple qubits, there is a unique extension which respects the tensor product:

$$\boldsymbol{\rho}^1 \otimes \boldsymbol{\rho}^2 \otimes \cdots \otimes \boldsymbol{\rho}^N = \boldsymbol{\rho} \quad \Leftrightarrow \quad \boldsymbol{\varrho} = \boldsymbol{\varrho}^1 \otimes \boldsymbol{\varrho}^2 \otimes \cdots \otimes \boldsymbol{\varrho}^N$$

Great Expectations

- From this, we may derive a simple formula for the expecation value of $\boldsymbol{\theta}$ directly from $\boldsymbol{\upsilon}$; for factorizable density operators, one has: $2^N\langle\boldsymbol{\rho\theta}\rangle$

$$= 2^{-N}\prod_{k=0}^{N}\prod_{\ell=0}^{N}\Big\langle\Big(\big(\boldsymbol{\varrho}^k+\boldsymbol{\varrho}^{\dagger k}\big)-\boldsymbol{\iota}\big(\boldsymbol{\varrho}^k-\boldsymbol{\varrho}^{\dagger k}\big)\Big)\cdots$$

$$\cdots\Big(\big(\boldsymbol{\vartheta}^\ell+\boldsymbol{\vartheta}^{\dagger\ell}\big)-\boldsymbol{\iota}\big(\boldsymbol{\vartheta}^\ell-\boldsymbol{\vartheta}^{\dagger\ell}\big)\Big)\Big\rangle$$

$$= 2^{-N}\prod_{k=0}^{N}\Big\langle\big(\boldsymbol{\varrho}^k+\boldsymbol{\varrho}^{\dagger k}\big)\big(\boldsymbol{\vartheta}^k+\boldsymbol{\vartheta}^{\dagger k}\big)-\big(\boldsymbol{\varrho}^k-\boldsymbol{\varrho}^{\dagger k}\big)\big(\boldsymbol{\vartheta}^k-\boldsymbol{\vartheta}^{\dagger k}\big)\Big\rangle$$

$$= \prod_{k=0}^{N}\Big\langle\boldsymbol{\varrho}^k\boldsymbol{\vartheta}^{\dagger k}+\boldsymbol{\varrho}^{\dagger k}\boldsymbol{\vartheta}^k\Big\rangle = 2^N\Big\langle\boldsymbol{\varrho\vartheta}^\dagger\Big\rangle = 2^N\Big\langle\boldsymbol{\varrho}^\dagger\boldsymbol{\vartheta}\Big\rangle$$

Equations of Motion

- Restricting ourselves to 2 qubits for simplicity, the usual density operator ρ is given in terms of the <u>reverse</u> even/odd parts of the parity-even as:

$$\varrho = \varrho_{\ominus} + \varrho_{\oplus} = \frac{1}{4} + \varrho_{\ominus} + \check{\varrho}_{\oplus} \quad \Rightarrow \quad \rho = \frac{1}{4} - \iota\varrho_{\ominus} - \check{\varrho}_{\oplus} = \frac{1}{2} - \iota\varrho_{\ominus} - \varrho_{\oplus}$$

- Then if we similarly decompose the parity-even Hamiltonian, the time-derivative of ρ is given by

$$\dot{\rho} = \iota[\rho, H] = -\iota[\varrho_{\ominus}, h_{\ominus}] - [\varrho_{\oplus}, h_{\ominus}] - [\varrho_{\ominus}, h_{\oplus}] + \iota[\varrho_{\oplus}, h_{\oplus}]$$

- It follows that ϱ evolves according to:

$$\dot{\varrho}_{\oplus} = [\varrho_{\oplus}, h_{\ominus}] + [\varrho_{\ominus}, h_{\oplus}], \quad \dot{\varrho}_{\ominus} = [\varrho_{\ominus}, h_{\ominus}] - [\varrho_{\oplus}, h_{\oplus}]$$

Results of Integration

- If $h_\oplus$ and $h_\ominus$ commute, the results of integrating these equations are:

$$\boldsymbol{\varrho}_\oplus(t) = e^{-t\boldsymbol{h}_\ominus}\Big(\cos(t\boldsymbol{h}_\oplus)\,\boldsymbol{\varrho}_\oplus\cos(t\boldsymbol{h}_\oplus)+\sin(t\boldsymbol{h}_\oplus)\,\boldsymbol{\varrho}_\oplus\sin(t\boldsymbol{h}_\oplus)\cdots$$

$$\cdots-\cos(t\boldsymbol{h}_\oplus)\,\boldsymbol{\varrho}_\ominus\sin(t\boldsymbol{h}_\oplus)+\sin(t\boldsymbol{h}_\oplus)\,\boldsymbol{\varrho}_\ominus\cos(t\boldsymbol{h}_\oplus)\Big)e^{t\boldsymbol{h}_\ominus}$$

$$\boldsymbol{\varrho}_\ominus(t) = e^{-t\boldsymbol{h}_\ominus}\Big(\cos(t\boldsymbol{h}_\oplus)\,\boldsymbol{\varrho}_\ominus\cos(t\boldsymbol{h}_\oplus)+\sin(t\boldsymbol{h}_\oplus)\,\boldsymbol{\varrho}_\ominus\sin(t\boldsymbol{h}_\oplus)\cdots$$

$$\cdots+\cos(t\boldsymbol{h}_\oplus)\,\boldsymbol{\varrho}_\oplus\sin(t\boldsymbol{h}_\oplus)-\sin(t\boldsymbol{h}_\oplus)\,\boldsymbol{\varrho}_\oplus\cos(t\boldsymbol{h}_\oplus)\Big)e^{t\boldsymbol{h}_\ominus}$$

- Letting $C_{\boldsymbol{h}}(t) \equiv \tfrac{1}{2}\Big(e^{t(\boldsymbol{h}_\ominus+i\boldsymbol{h}_\oplus)}+e^{-t(\boldsymbol{h}_\ominus+i\boldsymbol{h}_\oplus)}\Big)$

$$S_{\boldsymbol{h}}(t) \equiv \tfrac{i}{2}\Big(e^{t(\boldsymbol{h}_\ominus+i\boldsymbol{h}_\oplus)}-e^{-t(\boldsymbol{h}_\ominus+i\boldsymbol{h}_\oplus)}\Big)$$ ....

Integration (cont)

... the results for general h may be expressed as

$$\begin{aligned}
\varrho_{\oplus}(t) &= C_h(t)\,\varrho_{\oplus} C_h^{\dagger}(t) + S_h(t)\,\varrho_{\oplus} S_h^{\dagger}(t) \cdots \\
&\quad \cdots - C_h(t)\,\varrho_{\ominus} S_h^{\dagger}(t) + C_h(t)\,\varrho_{\ominus} S_h^{\dagger}(t) \\
\varrho_{\ominus}(t) &= C_h(t)\,\varrho_{\ominus} C_h^{\dagger}(t) + S_h(t)\,\varrho_{\ominus} S_h^{\dagger}(t) \cdots \\
&\quad \cdots + C_h(t)\,\varrho_{\oplus} S_h^{\dagger}(t) - C_h(t)\,\varrho_{\oplus} S_h^{\dagger}(t)
\end{aligned}$$

or, a little more compactly, as

$$\begin{aligned}
\varrho(t) &= C_h(t)\,\varrho\, C_h^{\dagger}(t) + S_h(t)\,\varrho\, S_h^{\dagger}(t) \cdots \\
&\quad \cdots + C_h(t)\,\varrho^{\dagger}\, S_h^{\dagger}(t) - C_h(t)\,\varrho^{\dagger}\, S_h^{\dagger}(t)
\end{aligned}$$

Entanglement Examples

- Let us define the standard quarter turns as
- $X_\pm^k \equiv \frac{1}{\sqrt{2}}\left(1 \pm \boldsymbol{\iota\sigma}_1^k\right),\;\; Y_\pm^k \equiv \frac{1}{\sqrt{2}}\left(1 \pm \boldsymbol{\iota\sigma}_2^k\right),\;\; Z_\pm^k \equiv \frac{1}{\sqrt{2}}\left(1 \pm \boldsymbol{\iota\sigma}_3^k\right)$
- The parity-even density operator of $|00\rangle$ is $Z_+^1 Z_+^2/2$. A Hadamard gate $W^1 \equiv (\boldsymbol{\iota\sigma}_1^1+\boldsymbol{\iota\sigma}_3^1)/\sqrt{2} = X_+^1 - Z_-^1$ maps it to $X_+^1 X_+^2/2$, to which we apply $\boldsymbol{h} = \pi\boldsymbol{\iota\sigma}_3^1\boldsymbol{\iota\sigma}_3^1/4$ with
- $C_{\boldsymbol{h}} = \cos\left(\frac{\pi}{4}\boldsymbol{\iota\sigma}_3^1\boldsymbol{\iota\sigma}_3^2\right) = \frac{1}{\sqrt{2}},\;\; S_{\boldsymbol{h}} = \sin\left(\frac{\pi}{4}\boldsymbol{\iota\sigma}_3^1\boldsymbol{\iota\sigma}_3^2\right) = \frac{1}{\sqrt{2}}\boldsymbol{\iota\sigma}_3^1\boldsymbol{\iota\sigma}_3^2$

  and get $\frac{1}{4}X_+^1X_+^2 + \frac{1}{4}\left(\boldsymbol{\iota\sigma}_3^1\boldsymbol{\iota\sigma}_3^2\right)X_+^1X_+^2\left(\boldsymbol{\iota\sigma}_3^1\boldsymbol{\iota\sigma}_3^2\right)$

$$+ \frac{1}{4}X_-^1X_-^2\left(\boldsymbol{\iota\sigma}_3^1\boldsymbol{\iota\sigma}_3^2\right) - \left(\boldsymbol{\iota\sigma}_3^1\boldsymbol{\iota\sigma}_3^2\right)\frac{1}{4}X_-^1X_-^2$$

Entanglement (cont)

- $= \frac{1}{4}\left(X_+^1 X_+^2 + X_-^1 X_-^2\right) + \frac{1}{4}\left(X_-^1 X_-^2 - X_+^1 X_+^2\right)\left(\boldsymbol{\iota\sigma}_3^1 \boldsymbol{\iota\sigma}_3^2\right)$
- $= \frac{1}{4}\left(1 + \boldsymbol{\iota\sigma}_1^1 \boldsymbol{\iota\sigma}_1^2\right) - \frac{1}{4}\left(\boldsymbol{\iota\sigma}_2^1 \boldsymbol{\iota\sigma}_3^2 + \boldsymbol{\iota\sigma}_3^1 \boldsymbol{\iota\sigma}_2^2\right)$
- Finally, a $-\pi/2$ rotation of the 2nd Qbit about the x-axis gives: $\Phi_+ = \frac{1}{4}\left(1 + \boldsymbol{\iota\sigma}_1^1 \boldsymbol{\iota\sigma}_1^2 - \boldsymbol{\iota\sigma}_2^1 \boldsymbol{\iota\sigma}_2^2 + \boldsymbol{\iota\sigma}_3^1 \boldsymbol{\iota\sigma}_3^2\right)$
- Similarly we can get the remaining Bell states:

$$\Phi_- = \frac{1}{4}\left(1 - \boldsymbol{\iota\sigma}_1^1 \boldsymbol{\iota\sigma}_1^2 + \boldsymbol{\iota\sigma}_2^1 \boldsymbol{\iota\sigma}_2^2 + \boldsymbol{\iota\sigma}_3^1 \boldsymbol{\iota\sigma}_3^2\right)$$

$$\Psi_+ = \frac{1}{4}\left(1 + \boldsymbol{\iota\sigma}_1^1 \boldsymbol{\iota\sigma}_1^2 + \boldsymbol{\iota\sigma}_2^1 \boldsymbol{\iota\sigma}_2^2 - \boldsymbol{\iota\sigma}_3^1 \boldsymbol{\iota\sigma}_3^2\right)$$

$$\Psi_- = \frac{1}{4}\left(1 - \boldsymbol{\iota\sigma}_1^1 \boldsymbol{\iota\sigma}_1^2 - \boldsymbol{\iota\sigma}_2^1 \boldsymbol{\iota\sigma}_2^2 - \boldsymbol{\iota\sigma}_3^1 \boldsymbol{\iota\sigma}_3^2\right)$$

The Partial Trace

- Some parity-even operator sum representations of the partial trace over a 0th Qbit are

$$2\langle \varrho \rangle^0 = X_+^0\left(Z_+^0 \varrho Z_-^0 + Z_-^0 \varrho Z_+^0\right)X_-^0 + X_-^0\left(Z_+^0 \varrho Z_-^0 + Z_-^0 \varrho Z_+^0\right)X_+^0$$

$$= \tfrac{1}{2}\left(\varrho - \iota\sigma_1^0 \varrho \iota\sigma_1^0 - \iota\sigma_2^0 \varrho \iota\sigma_2^0 - \iota\sigma_3^0 \varrho \iota\sigma_3^0\right)$$

- Applied to e.g. the GHZ state,
- $$\Xi = \iota\sigma_1^0 \iota\sigma_1^1 \iota\sigma_1^2\left(\tfrac{1}{2} - \tfrac{1}{\sqrt{8}}\left(Z_+^0 Z_+^1 Z_+^2 + Z_-^0 Z_-^1 Z_-^2\right)\right)$$
- $$\cdots + \tfrac{1}{\sqrt{8}}\left(Z_+^0 Z_+^1 Z_+^2 + Z_-^0 Z_-^1 Z_-^2\right)$$

this gives $2\langle \Xi \rangle^0 = \tfrac{1}{2}\left(Z_+^1 Z_+^2 + Z_-^1 Z_-^2\right)$ as expected.

Kraus Operator Sums

- The parity-even analog of a Kraus operator sum is:

$$
\begin{aligned}
\varrho'' &= \left\langle C_h \varrho C_h^\dagger + S_h \varrho S_h^\dagger + C_h \varrho S_h^\dagger - S_h \varrho C_h^\dagger \right\rangle^0 \\
&= -\sum_{\mu,\nu=0}^{3,3} \Bigg\langle \left\langle C_h \sqrt{\varrho^0}\, \iota\sigma_\mu^0 \right\rangle^0 \iota\sigma_\mu^0 \varrho' \iota\sigma_\nu^0 \left\langle C_h \sqrt{\varrho^0}\, \iota\sigma_\nu^0 \right\rangle^{0\dagger} + \cdots \\
&\qquad \left\langle S_h \sqrt{\varrho^0}\, \iota\sigma_\mu^0 \right\rangle^0 \iota\sigma_\mu^0 \varrho' \iota\sigma_\nu^0 \left\langle S_h \sqrt{\varrho^0}\, \iota\sigma_\nu^0 \right\rangle^{0\dagger} + \cdots \\
&\qquad \left\langle C_h \sqrt{\varrho^0}\, \iota\sigma_\mu^0 \right\rangle^0 \iota\sigma_\mu^0 \varrho' \iota\sigma_\nu^0 \left\langle S_h \sqrt{\varrho^0}\, \iota\sigma_\nu^0 \right\rangle^{0\dagger} - \cdots \\
&\qquad \left\langle S_h \sqrt{\varrho^0}\, \iota\sigma_\mu^0 \right\rangle^0 \iota\sigma_\mu^0 \varrho' \iota\sigma_\nu^0 \left\langle C_h \sqrt{\varrho^0}\, \iota\sigma_\nu^0 \right\rangle^{0\dagger} \Bigg\rangle^0 \\
&= -\sum_{\mu,\nu=0}^{3,3} \left\langle K_\mu \varrho K_\nu^\dagger + L_\mu \varrho L_\nu^\dagger + K_\mu \varrho L_\nu^\dagger - L_\mu \varrho K_\nu^\dagger \right\rangle^0 \left\langle \iota\sigma_\mu^0 \iota\sigma_\nu^0 \right\rangle \\
&= \sum_{\mu=0}^{3} \left\langle K_\mu \varrho K_\mu^\dagger + L_\mu \varrho L_\mu^\dagger + K_\mu \varrho L_\mu^\dagger - L_\mu \varrho K_\mu^\dagger \right\rangle^0
\end{aligned}
$$

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- A few references for those who are interested:

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  - My home page is "http://web.mit.edu/tfhavel/www"