

Geometric Algebra 2

Quantum Theory

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Spin

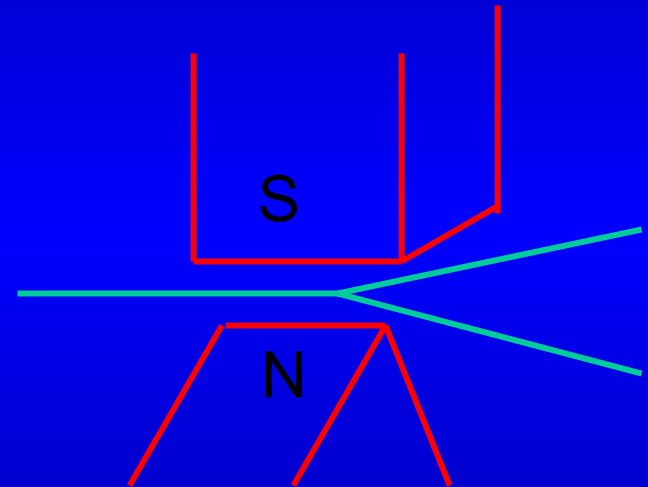
- Stern-Gerlach tells us that electron wavefunction contains two terms

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

- Describe state in terms of a **spinor**

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

- A 2-state system or **qubit**



Pauli Matrices

- Operators acting on a spinor must obey angular momentum relations

$$\hat{l}_i = -i\hbar\epsilon_{ijk}x_j\partial_k, \quad [\hat{l}_i, \hat{l}_j] = i\hbar\epsilon_{ijk}\hat{l}_k$$

- Get spin operators

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- These form a Clifford algebra
- A matrix representation of the geometric algebra of 3D space

Observables

- Want to place Pauli theory in a more geometric framework with $\hat{\sigma}_k \mapsto \boldsymbol{\sigma}_k$

- Construct **observables**

$$s_k = \frac{1}{2} \hbar n_k = \langle \psi | \hat{s}_k | \psi \rangle$$

- Belong to a unit vector
- Written in terms of polar coordinates, find parameterisation

$$|\psi\rangle = \begin{pmatrix} \cos(\theta/2)e^{-i\phi/2} \\ \sin(\theta/2)e^{i\phi/2} \end{pmatrix}$$

Rotors and Spinors

- From work on Euler angles, encode degrees of freedom in the **rotor**

$$R = e^{-\phi I \sigma_3 / 2} e^{-\theta I \sigma_2 / 2}$$

- Represent spinor / qubit as element of the **even subalgebra**:

$$|\psi\rangle = \begin{pmatrix} a^0 + ia^3 \\ -a^2 + ia^1 \end{pmatrix} \leftrightarrow \psi = a^0 + a^k I \sigma_k$$

- Verify that

$$\hat{\sigma}_k |\psi\rangle \leftrightarrow \sigma_k \psi \sigma_3 \quad (k = 1, 2, 3)$$

Keeps result in even algebra

Imaginary Structure

- Can construct imaginary action from

$$\hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

- So find that

$$i|\psi\rangle \leftrightarrow \sigma_1 \sigma_2 \sigma_3 \psi (\sigma_3)^3 = \psi I \sigma_3$$

- Complex structure controlled by a **bivector**
- Acts on the **right**, so **commutes** with operators applied to the left of the spinor
- Hints at a geometric substructure
- Can always use i to denote the structure

Inner Products

- The reverse operation in 3D is same as Hermitian conjugation
- Real part of inner product is

$$\mathbb{R}\langle\psi|\phi\rangle \leftrightarrow \langle\psi^\dagger|\phi\rangle$$

- Follows that full inner product is

$$\langle\psi|\phi\rangle \leftrightarrow \langle\psi^\dagger|\phi\rangle - \langle\psi^\dagger|\phi I\sigma_3\rangle I\sigma_3$$

- The projection onto the 1 and $I\sigma_3$ components,

$$\langle A \rangle_q = \frac{1}{2}(A + \sigma_3 A \sigma_3)$$

Observables

- Spin observables become

$$\langle \psi | \hat{\sigma}_k | \psi \rangle \leftrightarrow \langle \psi^\dagger \boldsymbol{\sigma}_k \psi \boldsymbol{\sigma}_3 \rangle = \boldsymbol{\sigma}_k \cdot (\psi \boldsymbol{\sigma}_3 \psi^\dagger)$$

- All information contained in the **spin vector**

$$\mathbf{s} = \frac{1}{2} \hbar \psi \boldsymbol{\sigma}_3 \psi^\dagger$$

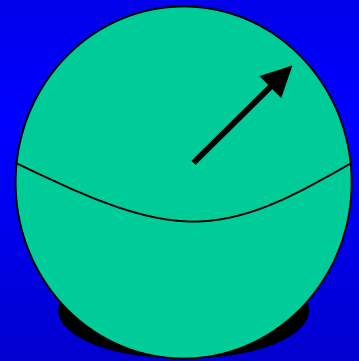
- Now define normalised rotor

$$\rho = \psi \psi^\dagger, \quad \psi = \rho^{1/2} R$$

- Operation of forming an observable reduces to

$$\mathbf{s} = \frac{1}{2} \hbar \rho \boxed{R \boldsymbol{\sigma}_3 R^\dagger}$$

← Same as classical expression



Rotating Spinors

- So have a natural ‘explanation’ for 2-sided construction of quantum observables
- Now look at composite rotations

$$R_\theta = \exp(-\hat{B}\theta/2), \quad R\sigma_3R^\dagger \mapsto R_\theta R\sigma_3R^\dagger R_\theta^\dagger$$

- So rotor transformation law is

$$R \mapsto R' = R_\theta R$$

- Take angle through to 2π

$$R' = e^{-\hat{B}\pi} R = -R$$

Sign change for fermions

Unitary Transformations

- Spinors can transform under the full unitary group $U(2)$
- Decomposes into $SU(2)$ and a $U(1)$ term
- $SU(2)$ term becomes a **rotor** on left
- $U(1)$ term applied on the right

$$U(\psi) = R\psi e^{\phi I \sigma_3}$$

- Separates out the group structure in a helpful way
- Does all generalise to multiparticle setting

Magnetic Field

- Rotor contained in $\psi = \rho^{1/2} R$
- Use this to Simplify equations
- Magnetic field $\hat{H} = -\frac{1}{2}\gamma \hbar B_k \hat{\sigma}_k$
- Schrodinger equation $\frac{d|\psi\rangle}{dt} = \frac{1}{2}\gamma i B_k \hat{\sigma}_k |\psi\rangle$
- Reduces to simple equation
$$\dot{\psi} = \frac{1}{2}\gamma B_k I \boldsymbol{\sigma}_k \psi = \frac{1}{2}\gamma I \mathbf{B} \psi$$
- Magnitude is constant, so left with **rotor equation**
$$\dot{R} = \frac{1}{2}\gamma I \mathbf{B} R$$

Density Matrices

- Mixed states are described by a **density matrix**
- For a pure state this is

$$\hat{\rho} = |\psi\rangle \langle\psi| = \begin{pmatrix} \alpha\alpha^* & \alpha\beta^* \\ \beta\alpha^* & \beta\beta^* \end{pmatrix}$$

- GA version is $\psi \frac{1}{2} (1 + \sigma_3) \psi^\dagger = \frac{1}{2}(1 + s)$
- Addition is fine in GA!
- General mixed state from sum

$$\rho = \frac{1}{2n}(n + s_1 + \cdots + s_n) = \frac{1}{2}(1 + P), \quad |P| \leq 1$$

Spacetime Algebra

- Basic tool for relativistic physics is the spacetime algebra or STA.

1 γ_μ $\gamma_\mu \gamma_\nu$ $I \gamma_\mu$ $I = \gamma_0 \gamma_1 \gamma_2 \gamma_3$
1 scalar 4 vectors 6 bivectors 4 trivectors 1 pseudoscalar

- Generators satisfy

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \text{diag}(+ - - -), \quad \mu = 0 \dots 3$$

- A **matrix-free** representation of Dirac theory
- Currently used for classical mechanics, scattering, tunnelling, supersymmetry, gravity and quantum information

Relative Space

- Determine 3D space relative for observer with velocity given by timelike vector γ_0
- Suppose event has position x in natural units

$$t = x \cdot \gamma_0 \quad \mathbf{x} = x \wedge \gamma_0$$
$$t + \mathbf{x} = x \gamma_0$$

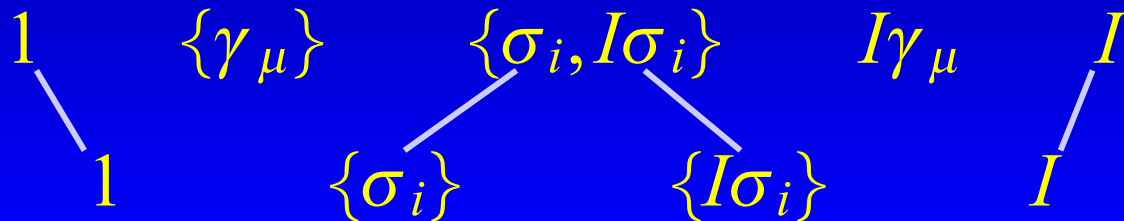
- The basis elements of relative vector are

$$\sigma_i = \gamma_i \gamma_0$$

- Satisfy
$$\begin{aligned} \sigma_i \cdot \sigma_j &= \frac{1}{2}(\gamma_i \gamma_0 \gamma_j \gamma_0 + \gamma_j \gamma_0 \gamma_i \gamma_0) \\ &= \frac{1}{2}(-\gamma_i \gamma_j - \gamma_j \gamma_i) = \delta_{ij} \end{aligned}$$

Relative Split

- Split bivectors with γ_0 to determine relative split



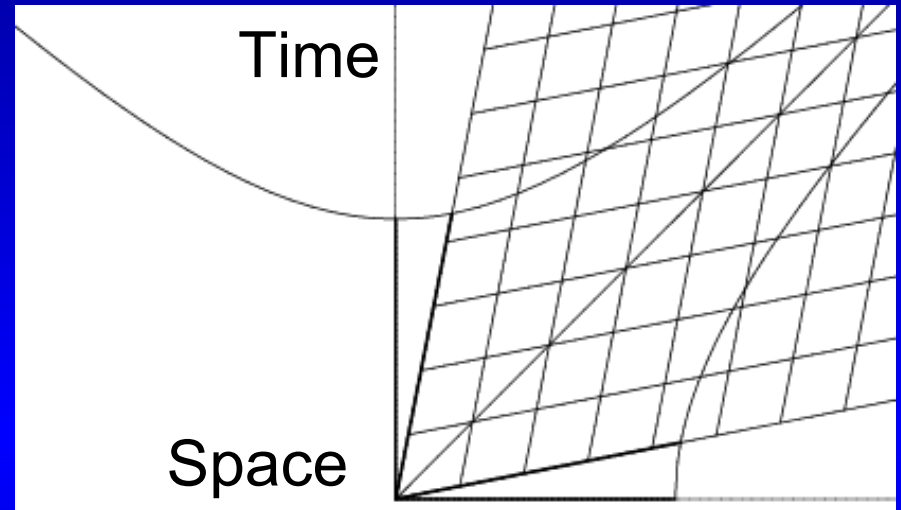
- Relative vectors generate 3D algebra with same volume element

$$\sigma_1 \sigma_2 \sigma_3 = (\gamma_1 \gamma_0)(\gamma_2 \gamma_0)(\gamma_3 \gamma_0) = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = I$$

- Relativistic (Dirac) spinors constructed from full 3D algebra

Lorentz Transformation

- Moving observers construct a new coordinate grid
- Both position and time coordinates change



$$t' = \gamma(t - \beta z), \quad z' = \gamma(z - \beta t),$$

$$\gamma = (1 - \beta^2)^{-1/2} \quad \beta = v/c$$

Need to re-express this in terms of **vector transformations**

Frames and Boosts

- Vector unaffected by coordinate system, so

$$x = x^\mu e_\mu = x^{\mu'} e'_{\mu'}$$

- Frame vectors related by

$$e'_0 = \gamma(e_0 + \beta e_3), \quad e'_3 = \gamma(e_3 + \beta e_0)$$

- Introduce the **hyperbolic angle**

$$\tanh(\alpha) = \beta$$

- Transformed vectors now

$$e'_0 = \cosh(\alpha) e_0 + \sinh(\alpha) e_3 = \exp(\alpha e_3 e_0) e_0$$

Exponential of a bivector

Spacetime Rotors

- Define the spacetime **rotor**

$$R = e^{\alpha e_3 e_0 / 2}$$

- A Lorentz transformation can now be written in rotor form

$$e'_\mu = R e_\mu \tilde{R}$$

- Use the tilde for reverse operation in the STA (dagger is frame-dependent)
- Same rotor description as 3D
- Far superior to 4X4 matrices!

Pure Boosts

- Rotors generate proper orthochronous transformations
- Suppose we want the pure boost from u to v

$$v = Lu\tilde{L} = L^2u$$

- Solution is

$$L = \frac{1 + vu}{[2(1 + u \cdot v)]^{1/2}} = \exp \left(\frac{\alpha}{2} \frac{v \wedge u}{|v \wedge u|} \right)$$

- Remainder of a general rotor is

$$R = LU, \quad U = \tilde{L}R$$

A 3D rotor

Velocity and Acceleration

- Write arbitrary 4-velocity as

$$v = R\gamma_0\tilde{R}$$

- Acceleration is

$$\dot{v} = \frac{d}{d\tau}(R\gamma_0\tilde{R}) = \dot{R}\gamma_0\tilde{R} + R\gamma_0\dot{\tilde{R}}$$

- But $\dot{R}\tilde{R} = -R\dot{\tilde{R}}$ Pure bivector

- So $\dot{v} = \dot{R}\tilde{R}v - v\dot{R}\tilde{R} = 2(\dot{R}\tilde{R}) \cdot v$

Acceleration bivector

The vector derivative

- Define the vector derivative operator in the standard way

$$\nabla = \sum_k e^k \frac{\partial}{\partial x^k}$$

- So components of this are **directional derivatives**
- But now the vector product terms are **invertible**
- Can construct Green's functions for ∇
- These are **Feynman propagators** in spacetime

2 Dimensions

- Vector derivative is

$$\nabla = e_1 \frac{\partial}{\partial x} + e_2 \frac{\partial}{\partial y}$$

- Now introduce the scalar+pseudoscalar field

$$\varphi = u + Iv$$

- Find that

$$\nabla \varphi = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) e_1 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) e_2$$

- Same terms that appear in the **Cauchy-Riemann** equations!

Analytic Functions

- Vector derivative closely related to definition of analytic functions

- Statement that φ is **analytic** is $\nabla\varphi = 0$

- Cauchy integral formula provides inverse

- This generalises to arbitrary dimensions

- Can construct power series in z because

$$\nabla z = \nabla(e_1 x) = \nabla(2e_1 \cdot r - x e_1) = 0$$

- Lose the commutativity in higher dimensions
- But this does not worry us now!

Spacetime Vector Derivative

- Define spacetime vector derivative

$$\nabla = \gamma^\mu \partial_\mu, \quad \partial_\mu = \frac{\partial}{\partial x^\mu}$$

- Has a spacetime split of the form

$$\nabla \gamma_0 = (\gamma^0 \partial_t + \gamma^i \partial_i) \gamma_0 = \partial_t - \boldsymbol{\sigma}_i \partial_i = \partial_t - \boldsymbol{\nabla}$$

- First application - Maxwell equations

$$\begin{aligned} \nabla \cdot \mathbf{D} &= \rho & \nabla \cdot \mathbf{B} &= 0 \\ -\nabla \times \mathbf{E} &= \frac{\partial}{\partial t} \mathbf{B} & \nabla \times \mathbf{H} &= \frac{\partial}{\partial t} \mathbf{D} + \mathbf{J} \end{aligned}$$

Maxwell Equations

- Assume no magnetisation and polarisation effects and revert to natural units
- Maxwell equations become, in GA form

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \rho & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \wedge \mathbf{E} &= -\partial_t(I\mathbf{B}) & \nabla \wedge \mathbf{B} &= I(\mathbf{J} + \partial_t\mathbf{E})\end{aligned}$$

- Naturally assemble equations for the divergence and (bivector) curl
- Combine using geometric product

$$\nabla(\mathbf{E} + I\mathbf{B}) + \partial_t(\mathbf{E} + I\mathbf{B}) = \rho - \mathbf{J}$$

STA Form

- Define the field strength (Faraday bivector)

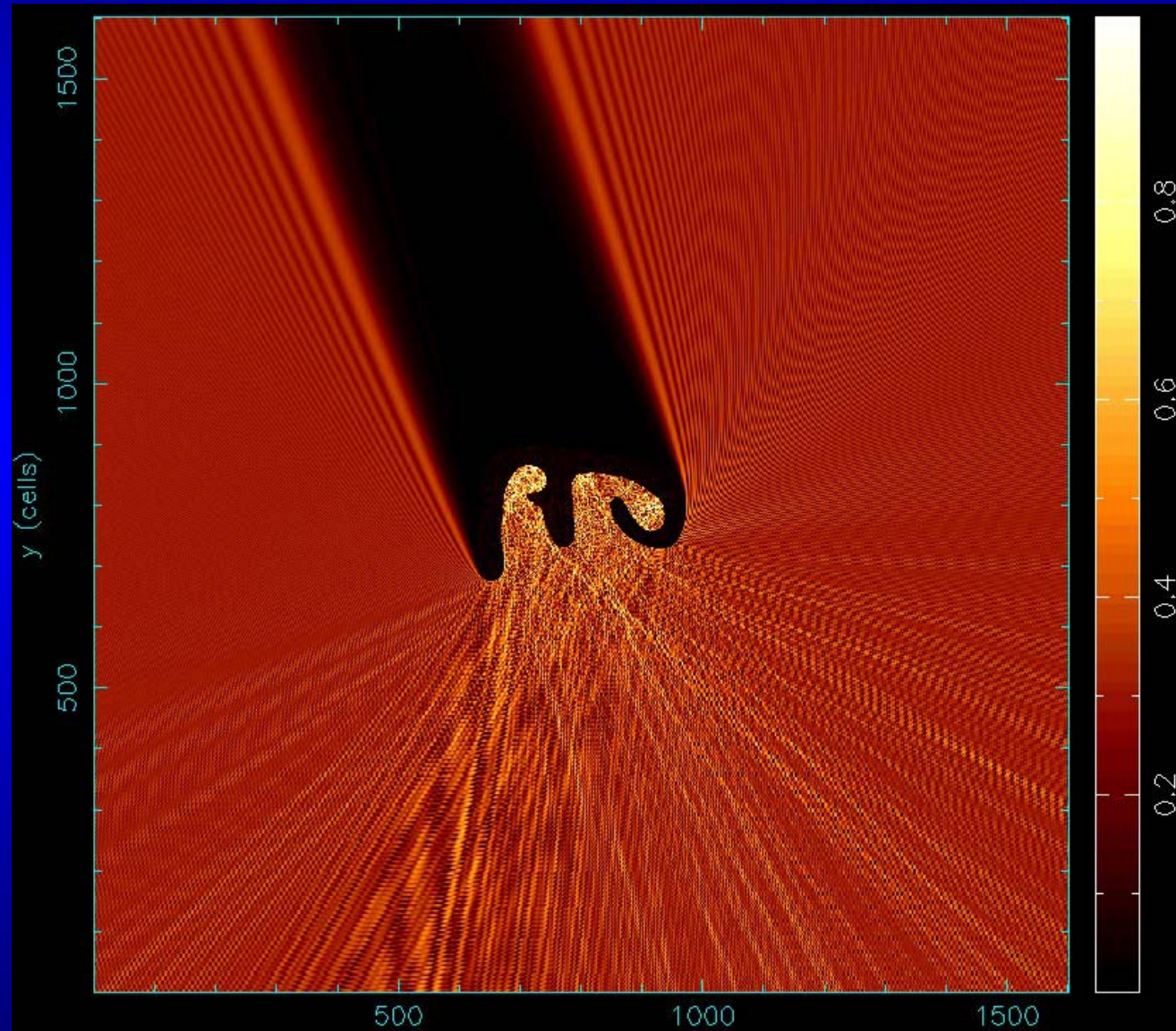
$$F = E + IB$$

- And current $J\gamma_0 = \rho + J$
- All 4 Maxwell equations unite into the single equation

$$\nabla F = J$$

- Spacetime vector derivative is **invertible**, can carry out first-order propagator theory
- First-order Green's function for scattering

Application



Lorentz Force Law

- Non-relativistic form is

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

- Can re-express in relativistic form as

$$\dot{v} = 2(\dot{R}\tilde{R}) \cdot v = \frac{q}{m} F \cdot v$$

- Simplest form is provided by rotor equation

$$\dot{R} = \frac{q}{2m} F R$$

Spin Dynamics

- Suppose that a particle carries a spin vector s along its trajectory

$$s \cdot v = 0, \quad s = R\gamma_3\tilde{R}$$

- Simplest form of rotor equation then has

$$\dot{s} = \frac{q}{m} F \cdot s$$

- Non relativistic limit to this equation is

$$\dot{s} = \frac{q}{m} s \times B$$

Equation for a particle
with $g=2$!

Exercises

- 2 spin-1/2 states are represented by ϕ and ψ , with accompanying spin vectors

$$\mathbf{s}_1 = \phi \boldsymbol{\sigma}_3 \tilde{\phi}, \quad \mathbf{s}_2 = \psi \boldsymbol{\sigma}_3 \tilde{\psi}$$

- Prove that

$$\frac{|\langle \phi | \psi \rangle|^2}{\langle \phi | \phi \rangle \langle \psi | \psi \rangle} = \frac{1}{2}(1 + \cos \theta) \quad \mathbf{s}_1 \cdot \mathbf{s}_2 = |\mathbf{s}_1| |\mathbf{s}_2| \cos(\theta)$$

- Given that

$$L = \frac{1 + vu}{[2(1 + v \cdot u)]^{1/2}}$$

- Prove that

$$v = Lu\tilde{L}$$