

Geometric Algebra 3

Dirac Theory and Multiparticle Systems

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Dirac Algebra

- Dirac matrix operators are

$$\hat{\gamma}_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}, \quad \hat{\gamma}_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- These act on 4-component Dirac spinors

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix}$$

- These spinors satisfy a **first-order** wave equation

$$i\hat{\gamma}^\mu \partial_\mu |\psi\rangle = m|\psi\rangle$$

STA Form

- Adapt the map for Pauli spinors

$$|\psi\rangle = \begin{pmatrix} |\phi\rangle \\ |\eta\rangle \end{pmatrix} \leftrightarrow \psi = \phi + \eta\sigma_3$$

- Action of the various operators now

$$\begin{aligned} \hat{\gamma}_\mu |\psi\rangle &\leftrightarrow \gamma_\mu \psi \gamma_0 \\ i |\psi\rangle &\leftrightarrow \psi I\sigma_3 \\ \hat{\gamma}_5 |\psi\rangle &\leftrightarrow \psi \sigma_3 \end{aligned}$$

Imaginary structure
still a bivector

Dirac equation $\nabla \psi I\sigma_3 - eA\psi = m\psi\gamma_0$

Comments

- Dirac equation based on the spacetime vector derivative
- Same as the Maxwell equation, so similar propagator structure
- Electromagnetic coupling from gauge principal
- Plane wave states have

$$p\psi = m\psi\gamma_0$$

A boost plus a rotation

Observables

- Observables are
 1. Gauge invariant
 2. Transform covariantly under Lorentz group $\psi \mapsto R\psi$

Bilinear covariant	Standard form	STA equivalent	Frame-free form
Scalar	$\langle \bar{\psi} \psi \rangle$	$\langle \psi \tilde{\psi} \rangle$	$\rho \cos(\beta)$
Vector	$\langle \bar{\psi} \hat{\gamma}_\mu \psi \rangle$	$\gamma_\mu \cdot (\psi \gamma_0 \tilde{\psi})$	$\psi \gamma_0 \tilde{\psi} = J$
Bivector	$\langle \bar{\psi} i \hat{\gamma}_{\mu\nu} \psi \rangle$	$(\gamma_\mu \wedge \gamma_\nu) \cdot (\psi I \sigma_3 \tilde{\psi})$	$\psi I \sigma_3 \tilde{\psi} = S$
Pseudovector	$\langle \bar{\psi} \hat{\gamma}_\mu \hat{\gamma}_5 \psi \rangle$	$\gamma_\mu \cdot (\psi \gamma_3 \tilde{\psi})$	$\psi \gamma_3 \tilde{\psi} = s$
Pseudoscalar	$\langle \bar{\psi} i \hat{\gamma}_5 \psi \rangle$	$\langle \psi \tilde{\psi} I \rangle$	$-\rho \sin(\beta)$

Current

- Main observable is the Dirac current

$$J = \psi \gamma_0 \tilde{\psi}$$

- Satisfies the conservation equation

$$\nabla \cdot J = 0$$

- Understand the observable better by writing

$$\psi \tilde{\psi} = \rho e^{I\beta} \quad \psi = \rho^{1/2} e^{I\beta/2} R$$

$$J = \rho R \gamma_0 \tilde{R}$$

 A Lorentz transformation

Current 2

- The Dirac current has a wider symmetry group than $U(1)$.
- Take $\psi \mapsto \psi M$
- Require $M\gamma_0\tilde{M} = \gamma_0$
- Four generators satisfy this requirement

$$I\sigma_1, \quad I\sigma_2, \quad I\sigma_3, \quad I$$

- Arbitrary transform

$$\psi \mapsto \psi e^{I\mathbf{b}} e^{I\phi}$$

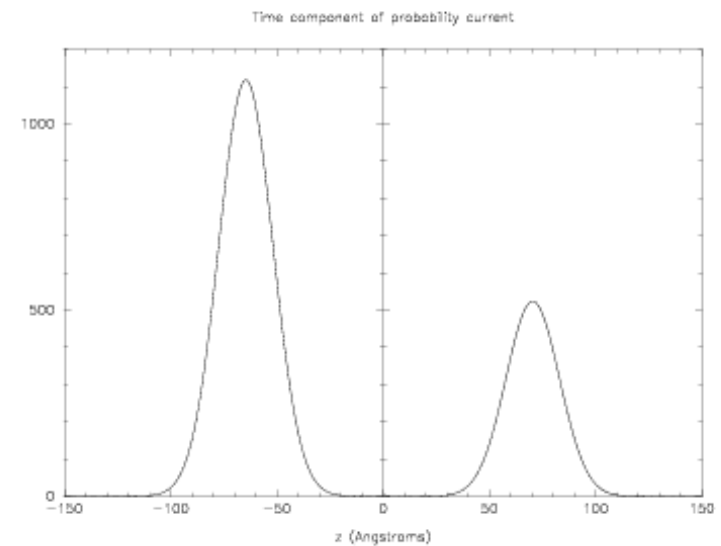
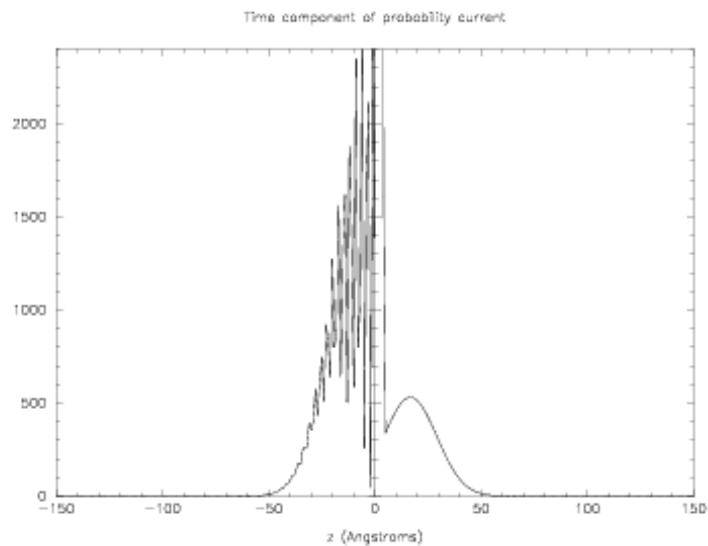
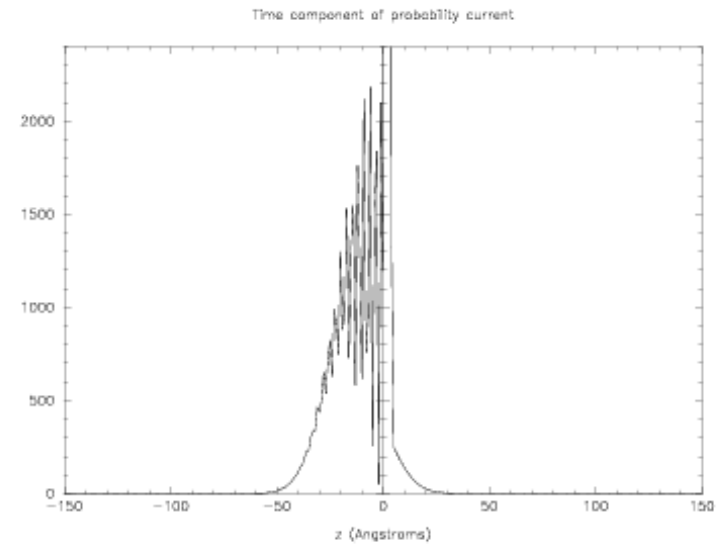
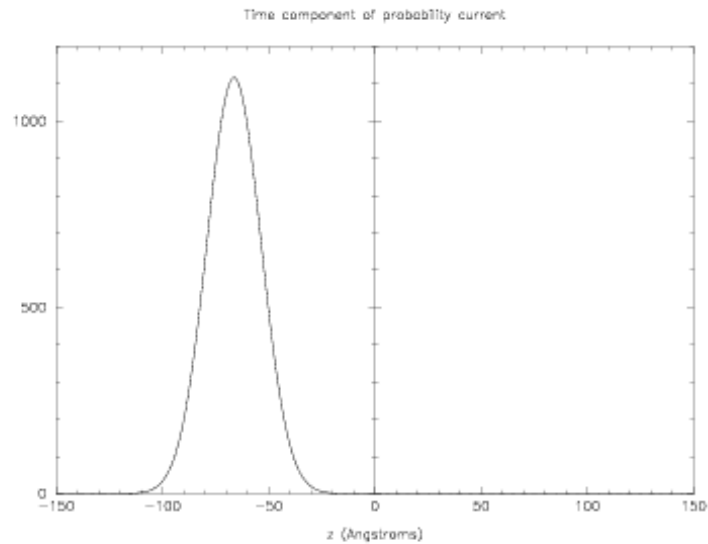
$SU(2)$

$U(1)$

Streamlines

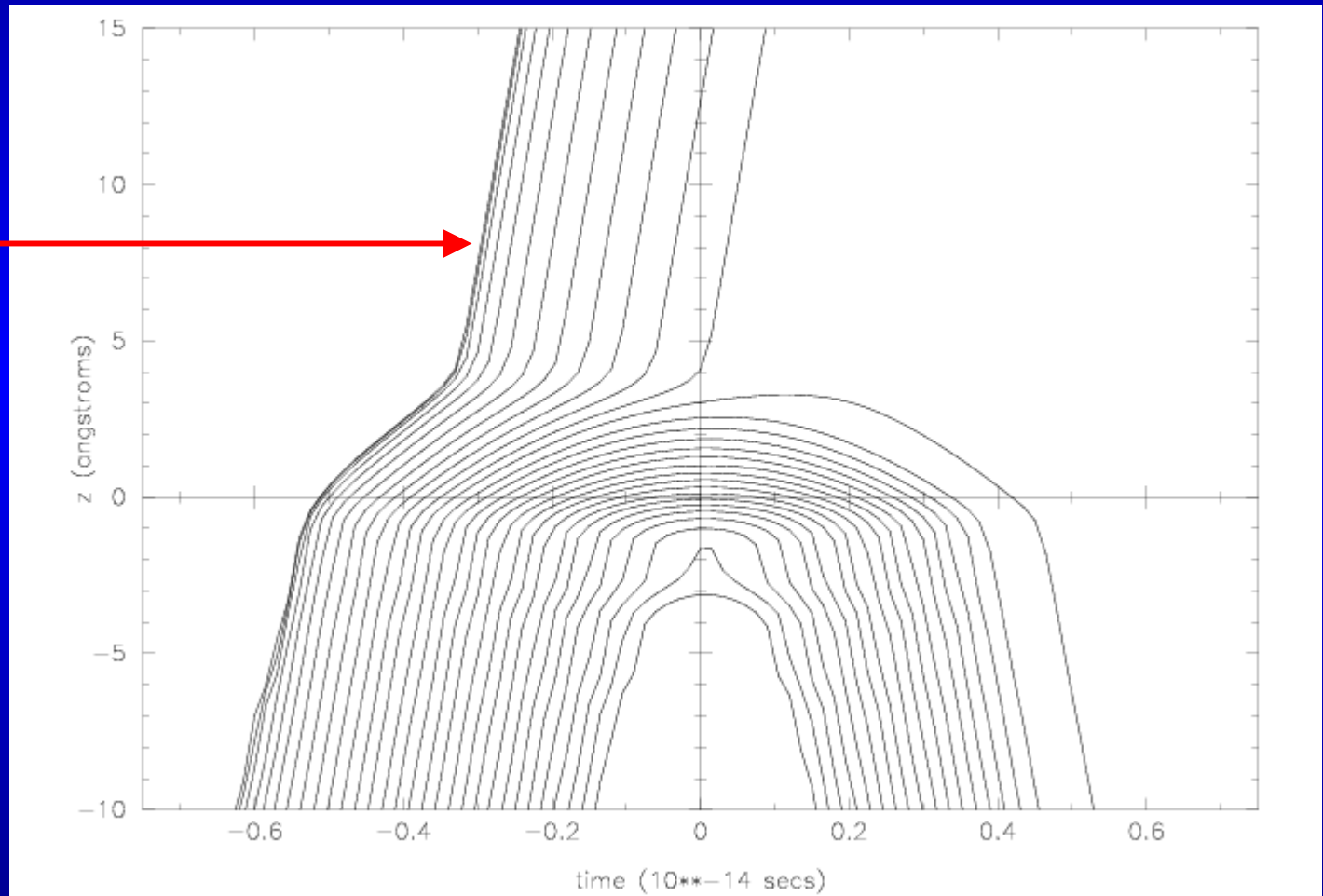
- The conserved current tells us where the probability density flows
- Makes sense to plot current streamlines
- These are genuine, local observables
- Not the same as following a Bohmian interpretation
- No need to insist that a 'particle' actually follows a given streamline
- Tunnelling is a good illustration

Tunnelling



Streamlines

Only front of
the packet
gets through



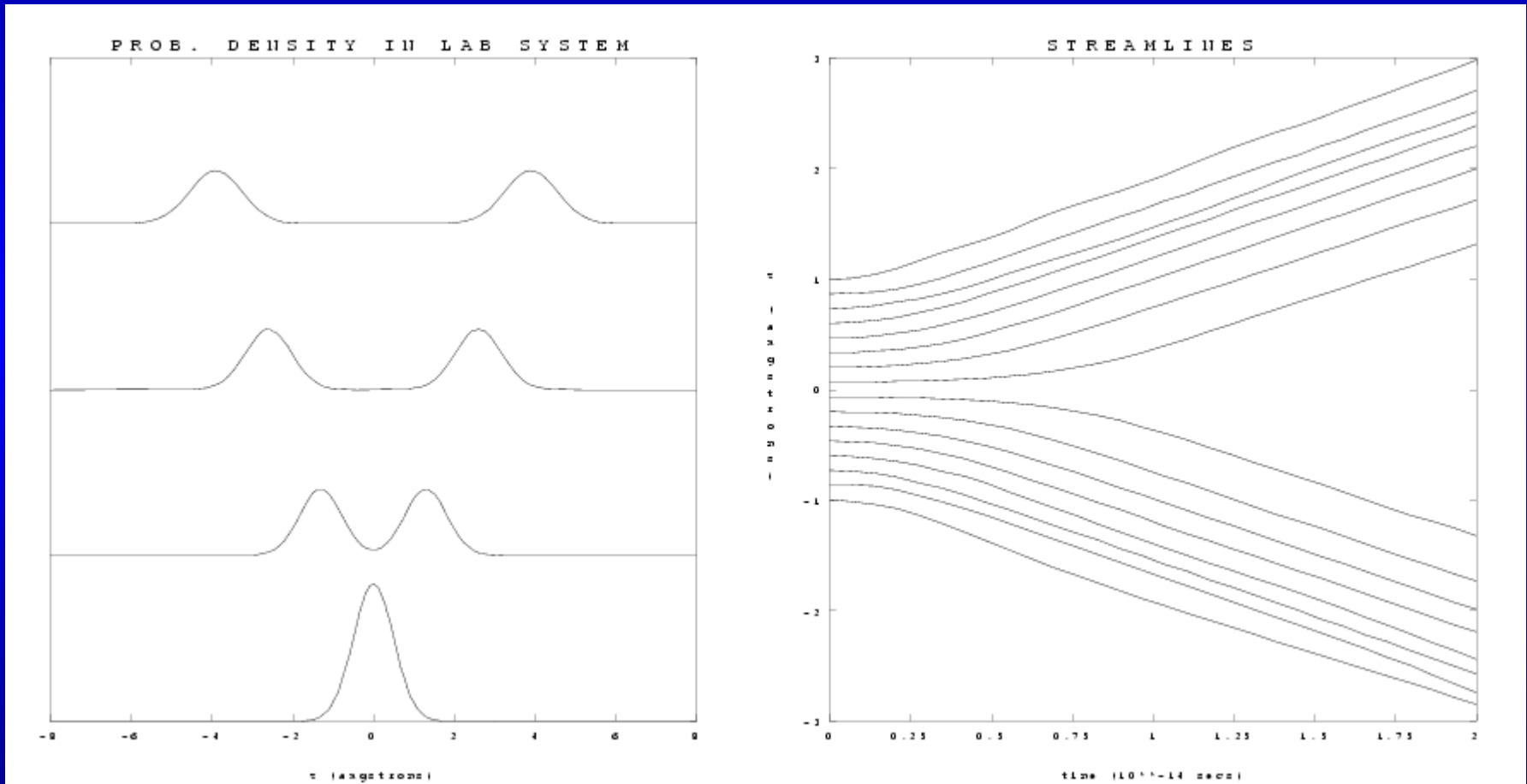
Spin Vector

- The Dirac spin observable is

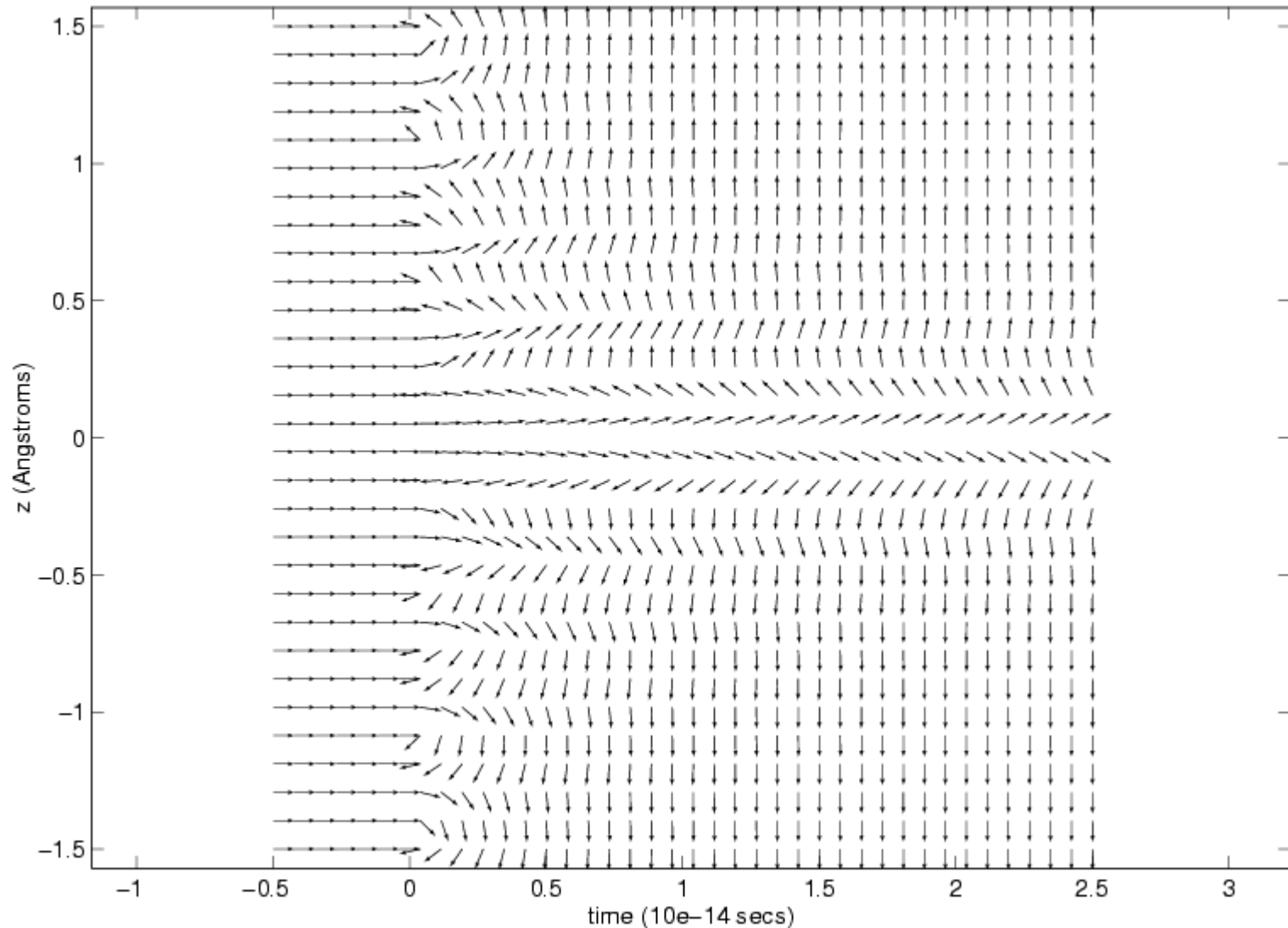
$$s = \psi \gamma_3 \tilde{\psi} = \rho R \gamma_0 \tilde{R}$$

- Same structure as used in classical model
- Use a 1D wavepacket to simulate a spin measurement
- Magnetic field simulated by a delta function shock
- Splits the initial packet into 2

Streamlines



Spin Orientation



Multiparticle Space

- Now suppose we want to describe n particles.
- View their trajectories as a path in $4n$ dimensional **configuration space**
- The vectors generators of this space satisfy

$$\gamma_{\mu}^a \gamma_{\nu}^b + \gamma_{\nu}^b \gamma_{\mu}^a = \begin{cases} 0 & a \neq b \\ 2\eta_{\mu\nu} & a = b \end{cases}$$

- Generators from different spaces **anticommute**
- These give a means of projecting out individual particle species

N-Particle Bivectors

- Now form the relative bivectors from separate spaces $\sigma_i^a = \gamma_i^a \gamma_0^a$

- These satisfy

$$\begin{aligned}\sigma_i^1 \sigma_j^2 &= \gamma_i^1 \gamma_0^1 \gamma_j^2 \gamma_0^2 = \gamma_i^1 \gamma_j^2 \gamma_0^2 \gamma_0^1 \\ &= \gamma_j^2 \gamma_0^2 \gamma_i^1 \gamma_0^1 = \sigma_j^2 \sigma_i^1\end{aligned}$$

- Bivectors from different spaces **commute**
- This is the GA implementation of the **tensor product**
- ‘Explains’ the nature of multiparticle Hilbert space

Complex Structure

- In quantum theory, states all share a single complex structure.

- So in GA, 2 particle quantum states must satisfy $\psi(I\sigma_3)^1 = \psi(I\sigma_3)^2$

$$\psi = -\psi(I\sigma_3)^1 (I\sigma_3)^2 = \psi \frac{1}{2} \left(1 - (I\sigma_3)^1 (I\sigma_3)^2 \right)$$

- Define the 2 particle correlator

$$E = \frac{1}{2} \left(1 - (I\sigma_3)^1 (I\sigma_3)^2 \right)$$

$$E^2 = E$$

2-Particle States

- Correlator ensures that 2-particle states have 8 real degrees of freedom
- A 2-particle direct-product state is

$$|\psi, \phi\rangle \leftrightarrow \psi^1 \phi^2 E$$

- Action of imaginary is

$$\begin{aligned} i|\psi, \phi\rangle &\leftrightarrow \psi^1 \phi^2 E(I\sigma_3)^1 = \psi^1 \phi^2 E(I\sigma_3)^2 \\ &= \psi^1 \phi^2 J \end{aligned}$$

- Complex structure now generated by J

$$J^2 = -E$$

Operators

- Action of 2-particle Pauli operators

$$\begin{aligned}\hat{\sigma}_k \otimes \hat{I} |\psi\rangle &\leftrightarrow -(I\sigma_k)^1 \psi J \\ \hat{\sigma}_k \otimes \hat{\sigma}_l |\psi\rangle &\leftrightarrow -(I\sigma_k)^1 (I\sigma_l)^2 \psi E \\ \hat{I} \otimes \hat{\sigma}_k |\psi\rangle &\leftrightarrow -(I\sigma_k^2) \psi J\end{aligned}$$

- Inner product

$$\langle\psi|\phi\rangle \leftrightarrow (\psi, \phi)_q = 2\langle\phi E\tilde{\psi}\rangle - 2\langle\phi J\tilde{\psi}\rangle i$$

Generic
symbol for
complex
structure



- Examples

$$\begin{aligned}\langle\psi| \hat{\sigma}_k \otimes \hat{I} |\psi\rangle &\leftrightarrow -2(I\sigma_k)^1 \cdot (\psi J\tilde{\psi}) \\ \langle\psi| \hat{\sigma}_j \otimes \hat{\sigma}_k |\psi\rangle &\leftrightarrow -2\left((I\sigma_j)^1 (I\sigma_k)^2\right) \cdot (\psi E\tilde{\psi})\end{aligned}$$

Density Matrices

- A normalised 2 particle density matrix can be expressed as

$$\hat{\rho} = \frac{1}{4} \left(\hat{I} \otimes \hat{I} + a_k \hat{\sigma}_k \otimes \hat{I} + b_k \hat{I} \otimes \hat{\sigma}_k + c_{jk} \hat{\sigma}_j \otimes \hat{\sigma}_k \right)$$

- So, for example

$$a_k = \text{tr} \left(\hat{\rho} (\hat{\sigma}_k \otimes \hat{I}) \right) = -2(I\sigma_k)^1 \cdot (\psi J \tilde{\psi})$$

- All of the information in the density matrix is held in the observables

$$\psi E \tilde{\psi} \quad \psi J \tilde{\psi}$$

Inner products and traces

- Can write the overlap probability as

$$P(\psi, \phi) = |\langle \psi | \phi \rangle|^2 = \text{tr}(\hat{\rho}_\psi \hat{\rho}_\phi)$$

- So have, for n-particle pure states

$$P(\psi, \phi) = 2^{n-2} \left(\langle (\psi E \tilde{\psi})(\phi E \tilde{\phi}) \rangle - \langle (\psi J \tilde{\psi})(\phi J \tilde{\phi}) \rangle \right)$$

- The **partial trace** operation corresponds to forming the observables, and throwing out terms
- Clearly see how this is removing information
- For mixed states, can correlate on pseudoscalar

Schmidt Decomposition

- General way to handle 2-particle states is to write as a matrix and perform an SVD

$$|\psi\rangle = e^{i\chi} \left(\cos\left(\frac{\alpha}{2}\right) e^{\frac{i\tau}{2}} \begin{pmatrix} \cos\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \\ \sin\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \end{pmatrix} \otimes \begin{pmatrix} \cos\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \\ \sin\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \end{pmatrix} \right. \\ \left. + \sin\left(\frac{\alpha}{2}\right) e^{\frac{-i\tau}{2}} \begin{pmatrix} \sin\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \\ -\cos\left(\frac{\theta_1}{2}\right) e^{\frac{i\phi_1}{2}} \end{pmatrix} \otimes \begin{pmatrix} \sin\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \\ -\cos\left(\frac{\theta_2}{2}\right) e^{\frac{i\phi_2}{2}} \end{pmatrix} \right)$$

- Only works for bipartite states

GA Form

- Define the local states / operators

$$R = \psi(\theta_1, \phi_1) e^{I\sigma_3\tau/4}, \quad S = \psi(\theta_2, \phi_2) e^{I\sigma_3\tau/4}$$

- Result of the Schmidt decomposition can now be written

$$\psi = \rho R^1 S^2 \left(\cos(\alpha/2) + \sin(\alpha/2) I\sigma_2^1 I\sigma_2^2 \right) e^{J\chi} E$$

Local unitaries

Entangling term

- Now have a form which generalises to arbitrary particles

3-Particle States

- The GHZ state is $\psi = \frac{1}{\sqrt{2}}(|000\rangle - |111\rangle)$

- GA equivalent $\psi = \exp(\pi/4 (I\sigma_2)^1 (I\sigma_2)^2 (I\sigma_2)^3)$

- The W-state is more interesting

$$|W\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle)$$

- Goes to $W = R^1 R^2 R^3 T_{12} T_{13} T_{23} T_{123}(\pi/4)$

$$R = e^{\pi/4 I\sigma_2}$$

$$T_{12} = \cos(\pi/12) + \sin(\pi/12)(I\sigma_2)^1 (I\sigma_2)^2$$

Singlet State

- An example of an entangled, or **non-local**, state is the 2-particle **singlet** state

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \leftrightarrow \psi = \frac{1}{\sqrt{2}}(I\sigma_2^1 - I\sigma_2^2)E$$

- This satisfies the identity

$$M^1_\epsilon = \tilde{M}^2_\epsilon$$

- Gives straightforward proof of invariance

$$R^1 R^2_\epsilon = R^1 \tilde{R}^1_\epsilon = \epsilon$$

- Observables are $2\epsilon E\tilde{\epsilon} = 1 + (I\sigma_k)^1 (I\sigma_k)^2$

Relativistic States

- All of the previous considerations extend immediately to relativistic states
- Can give physical definitions of entanglement for Dirac states
- Some disagreement on these issues in current literature
- Has been suggested that relative observers **disagree** on entanglement and purity
- More likely that an inappropriate definition has been adopted

Relativistic Singlet

- Can extend the non-relativistic state to one invariant under boosts as well

$$\eta = \varepsilon(1 - I^1 I^2)$$

- This satisfies

$$R^1 R^2 \eta = R^1 \tilde{R}^1 \eta = \eta$$

A Lorentz rotor

- This state plays an important role in GA versions of 2-spinor calculus and twistor theory

Multiparticle Dirac Equation

- Relativistic multiparticle quantum theory is a slippery subject!
- Ultimately, most issues sorted by QFT
- Can make some progress, though, e.g. with Pauli principle

$$I_P = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3, \quad \Gamma_\mu = \frac{1}{\sqrt{2}} (\gamma_\mu^1 + \gamma_\mu^2)$$

- Antisymmetrised state constructed via

$$\psi_-(x) = \psi(x) + I_P \psi(I_P x I_P) I_P$$

Current

- For equal mass particles, basic equation is

$$\nabla\psi J = m\psi(\gamma_0^1 + \gamma_0^2)$$

- Get a conserved current in **8D space**

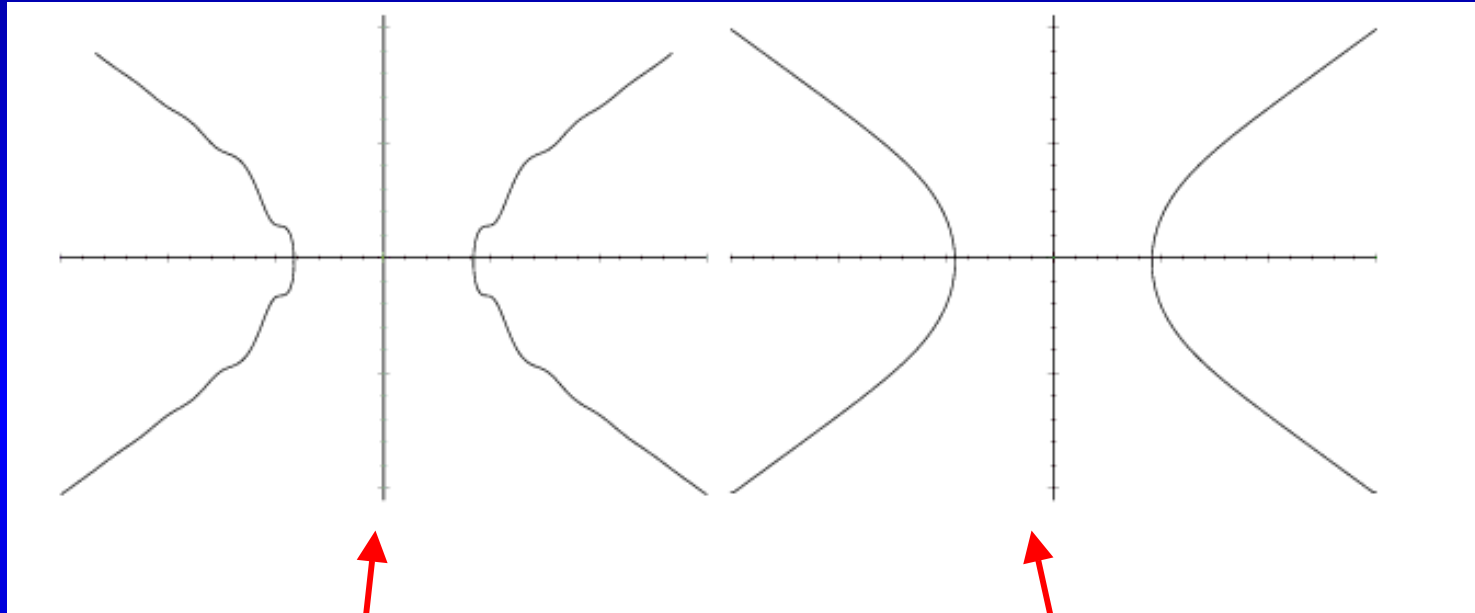
$$\mathcal{J} = \langle \psi(\gamma_0^1 + \gamma_0^2) \rangle_1$$

- Pauli principle ensures that

$$I_P \mathcal{J}(I_P x I_P) I_P = \mathcal{J}(x)$$

- Ensures that if 2 streamlines ever met, they could never separate

Plots



Spins aligned

Spins anti-aligned

In both cases the packets pass through each other

Exercises

- Verify that the overlap probability between 2 states is

$$P(\psi, \phi) = \frac{\langle (\psi E \tilde{\psi})(\phi E \tilde{\phi}) \rangle - \langle (\psi J \tilde{\psi})(\phi J \tilde{\phi}) \rangle}{2\langle \psi E \tilde{\psi} \rangle \langle \phi E \tilde{\phi} \rangle}$$

- Now suppose that one state is the singlet, and the other is separable. Prove that

$$P(\psi, \phi) = \langle \frac{1}{2}(1 - P^1 Q^2) \frac{1}{2}(1 + I\sigma_k^1 I\sigma_k^2) \rangle = \frac{1}{4}(1 - \cos \theta)$$

Angle between the spin vectors, or
between measuring apparatus