CrossMark

Dynamic Elastic Tides

Jack Wisdom[1] · **Jennifer Meyer[1,2]**

Abstract This is an exploration of dynamic tides on elastic bodies. The body is thought of as a dynamical system described by its modes of oscillation. The dynamics of these modes are governed by differential equations that depend on the rheology. The modes are damped by dissipation. Tidal friction occurs as exterior bodies excite the modes and the modes act back on the tide raising body. The whole process is governed by a closed set of differential equations. Standard results from tidal theory are recovered in a two-timescale approximation to the solution of these differential equations.

1 Introduction

Solar System studies of tides have, for the most part, been based directly or indirectly on the works of Darwin (1908). In these works the tide raising potential is Fourier expanded, and the response of the planet to each Fourier component is separately specified. The exterior potential of the tidally distorted planet takes the form of the tide raising potential, but it is phase delayed and given an amplitude proportional to the potential Love number, which encapsulates the structure of the planet. The large number of parameters is made tractable by choosing the phase delays to be proportional by a factor that has some particular frequency dependence. This frequency dependence has been either chosen for convenience or motivated by the behavior of materials at nontidal frequencies. For instance, if the tidal phase lags are taken to be proportional to frequency then the various tidal components form a single tidal bulge that is delayed in time from the tide raising potential (Mignard 1981; Hut 1981;

✉ Jack Wisdom
   wisdom@mit.edu

1   Massachusetts Institute of Technology, Cambridge, MA 02139, USA

2   Present Address: Clovis Community College, Fresno, CA 93730, USA

Eggleton et al. 1998). If the phase lags are taken to be independent of the frequency then we get the tidal models of Kaula (1966) and Goldreich (1963, 1966). There is scant knowledge of the actual frequency dependence of tides on planetary bodies. Williams (2008) determined the phase lags for the Moon at two frequencies. He reports a Q of approximately 30 for a forcing frequency of one month and a Q of about 35 for one year. Note that Williams actually measures the phase lag δ, and converts the phase lag to Q using the relation $1/Q = \tan \delta$ (Williams 2011, personal communication).

The method of Darwin (and those who follow him) is peculiar in that the process of raising tides is not described by a simple set of differential equations. If we want to calculate the tidal effect on an exterior body, we first let the body raise a set of tidal bulges (for each Fourier component), we impose a phase delay in the response by a specified amount for each component, then we calculate the effect of the phase delayed bulges on the exterior body. The end result may be a set of differential equations that model tidal friction, but the derivation of those equations is not through the solution of a closed set of differential equations.

We take a different approach, inspired by, but distinct from, the theory of dynamic tides on stars (Zahn 1975). We think of the planet as a dynamical system described by its modes of oscillation. The dynamics of the modes are determined by the internal distribution of density and the rheology. Dissipation occurs entirely in the modal dynamics, which may be more complicated than a simple second order differential equation modelling a damped harmonic oscillator, but is nevertheless an ordinary differential equation. Time varying tidal potentials excite the modes and the modes respond according to their own dynamics. In turn, the exterior potential of the modally distorted planet affects the tide raising body. The whole process is described by a closed set of differential equations. Multiple time scale analysis of these differential equations allow us to recover Darwin-like approximate expressions for tidal evolution and compare to standard results. Our framework is logically clean. The dynamics of the modes determines all the phase lags in the approximate expressions, but we need not go to these approximate expressions—we can directly integrate the full equations. Our goal is to develop a framework into which it will be possible to incorporate different rheologies in a logically clear and modular fashion. This paper describes the framework and works out the details for one particular model of dissipation in the modes. For simplicity we assume the body is homogeneous with uniform material properties.

The plan of the paper is as follows. In the next section we develop the expressions for the distortion of a body subject to an external tidal potential. We then develop the expressions for the tidal potential due to an exterior perturbing body. We then derive the free elastic modes of our body. This is followed by a decomposition of the tidal distortion in terms of the free elastic modes. We treat the modes as a dynamical system and next give the Lagrangian for the motion of the free modes. The Lagrangian for the motion of the planet with tidal distortion is developed in the following two sections. First, we develop expressions for the kinetic energy of a planet with modal distortion, then we develop the potential energy. The full Lagrangian is then formed and the equations of motion are developed. As a first application, we calculate the effect of elasticity on the frequency of the Chandler wobble. This is followed by a section on dissipation: we incorporate dissipation by modifying the differential equations governing the dynamics of the modes. This is the point at which one would incorporate different rheologies. We then calculate the rate of damping of the Chandler wobble. We come next to tidal friction. We do this in stages. We first calculate the rate of change of semimajor axis and rotation rate for a body in a circular orbit about a extended body with zero obliquity. We then extend these results to an eccentric orbit. We consider both the case where the rotation of the body is in spin-orbit resonance and where it is not. Then finally we extend the results to the general case with eccentricity and obliquity. This is followed by our concluding remarks.

2 Static tidal distortion

In this section we develop expressions for the distortion of a body subject to an external tidal potential. Our principal source for this section is Love (1944). We assume the body is elastic and incompressible.

The external disturbing potential is

$$U_T(x, y, z) = \sum_{lm} w_l^m \bar{X}_l^m (x/R, y/R, z/R),\tag{1}$$

where w_l^m are the coefficients of solid spherical harmonics $\bar{X}_l^m$ [see Eq. (303)], and R is the radius of the body. The external disturbing potential leads to a distortion of the body: this distortion contributes to the disturbing potential. The total potential can be written (Jefferys 1976) $V = \sum_{lm} V_l^m$ where

$$V_l^m(x, y, z) = w_l^m \bar{X}_l^m (x/R, y/R, z/R) + \frac{3g}{2l+1} \epsilon_l^m \bar{X}_l^m (x/R, y/R, z/R).\tag{2}$$

The second term arises from the potential of the distortion of the body. Given w_l^m we must solve for ϵ_l^m.

The displacement of a material particle in the body is $\mathbf{u}(t, x, y, z)$, where the coordinates (x, y, z) refer to the position in the body of material particle before displacement. The equations to be solved are

$$-\nabla p + \mu \nabla^2 \mathbf{u} + \rho \nabla V = 0.\tag{3}$$

Incompressibility implies

$$\nabla \cdot \mathbf{u} = 0.\tag{4}$$

Taking the divergence of the first equation and using the second equation, we see that $\nabla^2 p = 0$. We then put

$$p(x, y, z) = \rho V(x, y, z) + \tilde{p}(x, y, z),\tag{5}$$

where

$$\tilde{p}(x, y, z) = \sum_{lm} p_l^m \bar{X}_l^m (x/R, y/R, z/R),\tag{6}$$

and where $\bar{X}_l^m$ is a solid spherical harmonic. Let $\mathbf{u}_1$ be

$$\mathbf{u}_1(x, y, z) = \sum_{lm} \left(A_l^m r^2 \nabla \bar{X}_l^m (x/R, y/R, y/R) + B_l^m \mathbf{x} \bar{X}_l^m (x/R, y/R, y/R) \right),\tag{7}$$

where $\bar{X}_l^m$ is, again, a solid harmonic. Using Eqs. (313–315), we find that $\mathbf{u}_1$ of the form of Eq. (7) satisfies Eqs. (3) and (4) if the following two equations are satisfied:

$$p_l^m/\mu = (4l+2)A_l^m + 2B_l^m\tag{8}$$
$$0 = 2l A_l^m + (l+3)B_l^m,\tag{9}$$

respectively. For $l = 2$, these have the solution

$$A_2^m = \frac{5 p_2^m}{42\mu}\tag{10}$$

$$B_2^m = -\frac{4 p_2^m}{42\mu}.\tag{11}$$

To this solution we may add arbitrary solutions of the equation

$$\nabla^2 \mathbf{u} = 0, \tag{12}$$

with

$$\nabla \cdot \mathbf{u} = 0. \tag{13}$$

It will be enough to add solutions of the form

$$\mathbf{u}_2(x, y, z) = \sum_{lm} f_l^m \nabla \bar{X}_l^m (x/R, y/R, z/R). \tag{14}$$

The boundary conditions at the surface (see Love) require

$$0 = (l A_l^m + B_l^m) R^2 p_l^m + l f_l^m - R \epsilon_l^m \tag{15}$$

$$0 = (2l A_l^m + B_l^m) R^2 p_l^m + 2(l-1) f_l^m \tag{16}$$

$$\rho w_l^m = (\mu(2l A_l^m + (l+2) B_l^m) - 1) p_l^m + g\rho \left(1 - \frac{3}{2l+1}\right) \epsilon_l^m. \tag{17}$$

The solutions for $l = 2$ are

$$p_2^m = -(21/2)(\mu/gR) w_2^m / \Delta \tag{18}$$

$$f_2^m = 2(R/g) w_2^m / \Delta \tag{19}$$

$$\epsilon_2^m = (5/2)(w_2^m/g)/\Delta, \tag{20}$$

where

$$\Delta = 1 + 19\mu/(2\rho g R). \tag{21}$$

The displacement Love number is $h_2 = (5/2)/\Delta$. The total displacement $\mathbf{u}_1 + \mathbf{u}_2$, for this m, is

$$\mathbf{u}(x, y, z) = \mathbf{u}_T^m (x, y, z) h_2 w_2^m / (gR), \tag{22}$$

defining the tidal shape function $\mathbf{u}_T^m$. For $m = 0$, the tidal shape function $\mathbf{u}_T^0(0, 0, R)$ has the components $(0, 0, R)$, and the displacement at the surface along the z axis, the axis of maximum tidal distortion, is $h_2 w_2^0/g$.

3 Tidal disturbing potential

In this section we develop expressions for the tidal potential due to an external perturber. The tidal disturbing potential is the second harmonic contribution to the gravitational potential energy per unit mass

$$U_2(\mathbf{x}, \mathbf{R}') = -\frac{Gm}{(R')^3} \left(\frac{3}{2}(\mathbf{x} \cdot \hat{R}')^2 - \frac{1}{2}(\mathbf{x} \cdot \mathbf{x})\right), \tag{23}$$

where $\mathbf{x}$ has components (x, y, z), m is the mass of the perturbing, tide-raising body, and the $\mathbf{R}'$ is the vector from center of mass of the body to the tide raising body. The distance between the bodies is R', and the direction to the disturbing body is $\hat{R}'$. Let α, β, and γ be the direction cosines of $\hat{R}' = \alpha \hat{x} + \beta \hat{y} + \gamma \hat{z}$. Then, let N be the rotation that takes the direction $\hat{R}'$ to the $\hat{z}$ direction: $\hat{z} = N\hat{R}'$. Let $N(\theta, \psi) = M_x(\theta) M_z(\psi)$, where M_i are

rotations about the indicated axes. Then, $\theta = \mathrm{acos}(\gamma)$ and $\psi = \mathrm{atan}(\alpha, \beta)$.[1] Using the property $(N\mathbf{x}) \cdot (N\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$, we see

$$U_2(\mathbf{x}, \mathbf{R}') = U_2(N\mathbf{x}, \mathbf{z}) \tag{24}$$

$$= \sum_m w_2^m(\alpha, \beta, \gamma) \bar{X}_2^m(x/R, y/R, z/R). \tag{25}$$

We find $w_2^m(\alpha, \beta, \gamma) = W \bar{X}_2^m(\alpha, \beta, \gamma)$, where $W = -(Gm/R')(R/R')^2$. The corresponding tidal shape function can be obtained with an appropriate rotation of $\mathbf{u}_T^0$. We have

$$(N(\alpha, \beta, \gamma))^{-1} \mathbf{u}_T^0(N(\alpha, \beta, \gamma)\mathbf{x}) = \sum_m \bar{X}_2^m(\alpha, \beta, \gamma) \mathbf{u}_T^m(\mathbf{x}). \tag{26}$$

4 Elastic free modes

Next we develop expressions for the free modes of oscillation. Our principal source for this section is Lamb (1882). In his terminology modes corresponding to tidal distortions are vibrations in the second class. We will assume the material is incompressible. Our notation differs slightly from his.

The displacement is described as a sum over modes. The displacement is

$$\mathbf{u}(t, x, y, z) = \sum_{lmn} R A_{ln}^m \phi_{ln}^m(x, y, z) \cos(\omega_l^n t + \epsilon_{ln}^m), \tag{27}$$

where A_{ln}^m and ϵ_{ln}^m are the amplitude and phase of each mode. The modes of interest in the tidal problem are $l = 2$, m runs from $-l$ to l, and $n = 1, 2, \ldots$. The amplitude A_{ln}^m and the function ϕ_{ln}^m are dimensionless.

Let

$$\phi_{ln}^m(x, y, z)/R \tag{28}$$

$$= (\Upsilon_{ln} + \psi_{l-1}^m(\kappa_{ln}r)) \nabla \bar{X}_l^m - \frac{l}{l+1} \frac{\kappa_{ln}^2 r^{2l+3}}{(2l+1)(2l+3)} \psi_{l+1}(\kappa_{ln}r) \nabla \left(\frac{\bar{X}_l^m}{r^{2l+1}} \right), \tag{29}$$

where

$$\psi_n(\theta) = (-1)^n (2n+1)!! \left(\frac{1}{\theta} \frac{d}{d\theta} \right)^n \left(\frac{\sin\theta}{\theta} \right), \tag{30}$$

where κ_{ln} and Υ_{ln} are explained below. The functions ψ_n are related to the spherical Bessel functions.

The modal frequencies are determined by the condition

$$a_l d_l - b_l c_l = 0, \tag{31}$$

where, in the incompressible case,

$$a_l = \frac{(\kappa_{ln} R)^2}{2l+1} - 2(l-1) \tag{32}$$

$$b_l = 1 \tag{33}$$

---

[1] We use the two argument arctangent with the property $\psi = \mathrm{atan}(\sin(\psi), \cos(\psi))$, rather than the one argument arctangent with the property $\psi = \mathrm{atan}(\tan\psi)$, in order to retain quadrant information.

$$c_l = 2(l-1)\psi_{l-1}(\kappa_{ln}R) - \frac{(\kappa_{ln}R)^2}{2l+1}\psi_l(\kappa_{ln}R) \tag{34}$$

$$d_l = \frac{l}{l+1}\left(\psi_l(\kappa_{ln}R) + \frac{2(l+2)}{\kappa_{ln}R}\psi_l'(\kappa_{ln}R)\right) \tag{35}$$

where $\psi'(x) = d\psi(x)/dx$. Note that κ_{ln} occurs only in the combination $\kappa_{ln}R$, determined by Eq. (31). The frequencies ω_{ln} of the modes satisfy

$$\kappa_{ln}^2 = \omega_{ln}^2\rho/\mu, \tag{36}$$

where ρ is the density and μ is the rigidity. Numerically, we find the lowest frequency tidal mode has $\kappa_{21}R/\pi = 0.8484938956$, the next lowest frequency mode has $\kappa_{22}R/\pi = 1.7421226796$, and then $\kappa_{23}R/\pi = 2.8257142846$. Finally,

$$\Upsilon_{ln} = \frac{(\kappa_{ln}R)^2\psi_l(\kappa_{ln}R) - 2(l-1)(2l+1)\psi_{l-1}(\kappa_{ln}R)}{(\kappa_{ln}R)^2 - 2(l-1)(2l+1)}. \tag{37}$$

5 Representation of tidal distortion by elastic modes

Define the overlap integral

$$\langle \mathbf{u}_1, \mathbf{u}_2 \rangle = \frac{1}{V}\int_V (\mathbf{u}_1 \cdot \mathbf{u}_2)dV \tag{38}$$

where $V = (4/3)\pi R^3$, the volume of the body. The elastic modes have zero overlap. Let

$$(\beta_{2n})^2 = \langle \phi_{2n}^m, \phi_{2n}^m \rangle. \tag{39}$$

Note that β_{2n} is independent of m. We find: $\beta_{21} = 0.5325432017$, $\beta_{22} = 0.2145631038$, and $\beta_{23} = 0.0826504258$. Define the $l = 2$ normalized elastic modes:

$$\mathbf{u}_{nm}^M = R\,\phi_{2n}^m \big/ \beta_{2n}. \tag{40}$$

The tidal shape function $\mathbf{u}_T^0$ can be expanded in terms of the normalized elastic modes with $m = 0$. Let

$$\mathbf{u}_T^0(x, y, z) = \sum_n g_n\mathbf{u}_{n0}^M(x, y, z), \tag{41}$$

then

$$g_n = \langle \mathbf{u}_T^0, \mathbf{u}_n \rangle. \tag{42}$$

We find $g_1 = 0.5608256130$, $g_2 = -0.0381757369$, and $g_3 = 0.0039974227$. The tidal shape function and its representation in terms of the elastic modes is shown in Fig. 1.

If the tidal distortion is rotated, then the representing modes rotate accordingly. For a disturbing body with direction cosines (α, β, γ) the tidal distortion is

$$\mathbf{u}(x, y, z) = \frac{h_2 W}{gR}\sum_m \bar{X}_2^m(\alpha, \beta, \gamma)\mathbf{u}_T^m(x, y, z) \tag{43}$$

$$= \frac{h_2 W}{gR}\sum_{mn} g_n\bar{X}_2^m(\alpha, \beta, \gamma)\mathbf{u}_{nm}^M(x, y, z). \tag{44}$$

Fig. 1 The *solid line* shows the z component of the tidal shape function $\mathbf{u}_T^0$. The *dashed line* shows the representation of the tidal shape function using the $n = 1$ elastic mode. The *dotted line* shows the representation using the $n = 1$ and $n = 2$ modes. The three mode representation is indistinguishable from the *solid line* on this scale
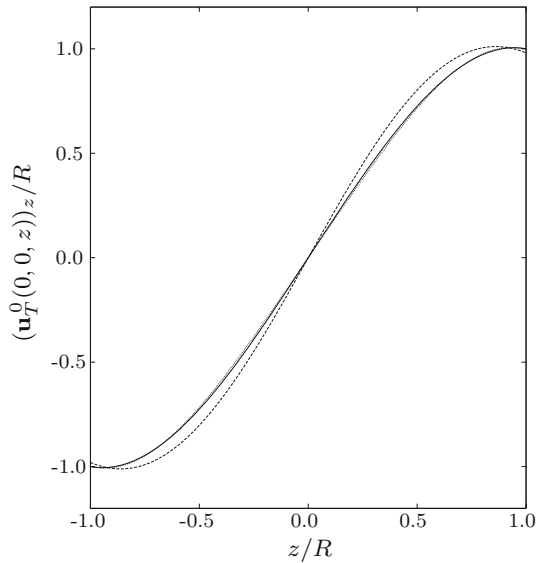
6 Lagrangian for elastic modes

We treat each mode as a degree of freedom. We assume here that the configuration of the body is given as a sum of modal distortions (with $l = 2$). We write the displacement as

$$\mathbf{u}(t, x, y, z) = \sum_{mn} \Pi_{nm}(t) \mathbf{u}_{nm}^M(x, y, z) \tag{45}$$

where $\Pi_{nm}(t)$ is the dimensionless modal coordinate at time t.

The kinetic energy in each mode is

$$T_{nm}(t, \Pi_{nm}, \dot{\Pi}_{nm}) = \frac{1}{2} MR^2 \dot{\Pi}_{nm}^2. \tag{46}$$

with $M = \rho V$, the mass of the body. Recall that the modes are normalized. We know the frequency of each mode, so we can write the elastic potential energy for each mode:

$$V_{nm}(t, \Pi_{nm}, \dot{\Pi}_{nm}) = \frac{1}{2} MR^2 \omega_{2n}^2 \Pi_{nm}^2 \tag{47}$$

The Lagrangian for the free elastic modes is then

$$L(t, \Pi, \dot{\Pi}) = \frac{1}{2} MR^2 \sum_{mn} \left(\dot{\Pi}_{nm}^2 - \omega_{2n}^2 \Pi_{nm}^2 \right). \tag{48}$$

This gives the equation of motion

$$MR^2 (\ddot{\Pi}_{nm} + \omega_{2n}^2 \Pi_{nm}) = 0, \tag{49}$$

confirming a free oscillation with frequency ω_{2n}. The modes will be forced when the full potential energy is developed.

7 Kinetic energy

The next task is to develop the kinetic energy of the rotating body, taking into account the fact that the shape is changing. We assume the undistorted starting configuration of the body is a sphere of radius R and density ρ, with moment of inertia $(2/5)MR^2$. The configuration of the body at time t is obtained by distorting and rotating this reference body. We give the body a distortion $\mathbf{u}_A$ that gives the principal moments of inertia and a time-dependent modal tidal distortion $\mathbf{u}_T(t)$, followed by a time-dependent rotation $M(t)$ in space. The position of each constituent is

$$\mathbf{x}_\alpha(t) = M(t)(\mathbf{x}_0 + \mathbf{u}_A + \mathbf{u}_T(t)). \tag{50}$$

The velocity of the constituent is

$$\dot{\mathbf{x}}_\alpha(t) = \dot{M}(t)(\mathbf{x}_0 + \mathbf{u}_A + \mathbf{u}_T(t)) + M(t)(\dot{\mathbf{u}}_T(t)) \tag{51}$$

$$= \dot{M}(t)M(t)^{-1}M(t)(\mathbf{x}_0 + \mathbf{u}_A + \mathbf{u}_T(t)) + M(t)(\dot{\mathbf{u}}_T(t)) \tag{52}$$

$$= \boldsymbol{\omega}(t) \times (M(t)(\mathbf{x}_0 + \mathbf{u}_A + \mathbf{u}_T(t))) + M(t)(\dot{\mathbf{u}}_T(t)) \tag{53}$$

$$= M(t)(\boldsymbol{\omega}'(t) \times (\mathbf{x}_0 + \mathbf{u}_A + \mathbf{u}_T(t)) + \dot{\mathbf{u}}_T(t)) \tag{54}$$

The square of the velocity is

$$(\dot{\mathbf{x}}_\alpha(t))^2 = (\boldsymbol{\omega}'(t) \times (\mathbf{x}_0 + \mathbf{u}_A)) \cdot (\boldsymbol{\omega}'(t) \times (\mathbf{x}_0 + \mathbf{u}_A)) \tag{55}$$

$$+ 2(\boldsymbol{\omega}'(t) \times (\mathbf{x}_0 + \mathbf{u}_A)) \cdot (\boldsymbol{\omega}'(t) \times \mathbf{u}_T(t) + \dot{\mathbf{u}}_T(t)) \tag{56}$$

$$+ (\boldsymbol{\omega}'(t) \times \mathbf{u}_T(t) + \dot{\mathbf{u}}_T(t)) \cdot (\boldsymbol{\omega}'(t) \times \mathbf{u}_T(t) + \dot{\mathbf{u}}_T(t)) \tag{57}$$

The distortion $\mathbf{u}_A$ that gives the principal moments is volume preserving; it can be represented as a gradient

$$\mathbf{u}_A(x, y, z) = aR^2 \nabla \bar{X}_2^0(x/R, y/R, z/R) + bR^2 \nabla \bar{X}_2^2(x/R, y/R, z/R), \tag{58}$$

where

$$a = \frac{(B+A)/2 - C}{A+B+C} \tag{59}$$

$$b = \frac{(\sqrt{3}/2)(B-A)}{A+B+C}. \tag{60}$$

The sum of the principal moments is $3I$ where $I = (2/5)MR^2$, the moment of inertia of a homogeneous sphere. In these expressions we are ignoring second order contributions to the moments in a and b; we are assuming a and b are small. Note that we can write

$$R\nabla \bar{X}_2^0 = (-\bar{X}_1^1, -\bar{X}_1^{-1}, 2\bar{X}_1^0). \tag{61}$$

and

$$R\nabla \bar{X}_2^2 = (\sqrt{3}\,\bar{X}_1^1, -\sqrt{3}\,\bar{X}_1^{-1}, 0). \tag{62}$$

For convenience, introduce

$$\mathbf{u}_A^0(x, y, z) = (x, y, z) \tag{63}$$

$$\mathbf{u}_A^1(x, y, z) = R^2 \nabla \bar{X}_2^0(x/R, y/R, z/R) \tag{64}$$

$$\mathbf{u}_A^2(x, y, z) = R^2 \nabla \bar{X}_2^2(x/R, y/R, z/R). \tag{65}$$

The first term in the kinetic energy integral is, by design,

$$\frac{1}{2} \int_V \rho((\boldsymbol{\omega}'(t) \times (\mathbf{x}_0 + \mathbf{u}_A)) \cdot (\boldsymbol{\omega}'(t) \times (\mathbf{x}_0 + \mathbf{u}_A))) dV \tag{66}$$

$$= \frac{1}{2} \left(A(\omega^a)^2 + B(\omega^b)^2 + C(\omega^c)^2 \right), \tag{67}$$

where the components of $\boldsymbol{\omega}'$ are $(\omega^a, \omega^b, \omega^c)$.

Introduce the integrals

$$I^{(1)}_{knmij} = \frac{1}{V} \int_V (e_i \times \mathbf{u}_A^k) \cdot (e_j \times \mathbf{u}_{nm}^M) dV, \tag{68}$$

$$I^{(2)}_{knmi} = \frac{1}{V} \int_V (e_i \times \mathbf{u}_A^k) \cdot \mathbf{u}_{nm}^M dV, \tag{69}$$

$$I^{(3)}_{nmn'm'ij} = \frac{1}{V} \int_V (e_i \times \mathbf{u}_{nm}^M) \cdot (e_j \times \mathbf{u}_{n'm'}^M) dV, \tag{70}$$

$$I^{(4)}_{nmn'm'i} = \frac{1}{V} \int_V (e_i \times \mathbf{u}_{nm}^M) \cdot \mathbf{u}_{n'm'}^M dV. \tag{71}$$

In terms of these integrals, the complete kinetic energy is

$$T(t; q, \Pi; \omega', \dot{\Pi}) = \frac{1}{2} \left(A(\omega^a)^2 + B(\omega^b)^2 + C(\omega^c)^2 \right) \tag{72}$$

$$+ MR^2 \sum_{knmij} a_k I^{(1)}_{knmij} (\omega')^i (\omega')^j \Pi_{nm} \tag{73}$$

$$+ MR^2 \sum_{knmi} a_k I^{(2)}_{knmi} (\omega')^i \dot{\Pi}_{nm} \tag{74}$$

$$+ \frac{1}{2} MR^2 \sum_{nmn'm'ij} I^{(3)}_{nmn'm'ij} (\omega')^i (\omega')^j \Pi_{nm} \Pi_{n'm'} \tag{75}$$

$$+ MR^2 \sum_{nmn'm'i} I^{(4)}_{nmn'm'i} (\omega')^i \Pi_{nm} \dot{\Pi}_{n'm'} \tag{76}$$

$$+ \frac{1}{2} MR^2 \sum_{nm} \left(\dot{\Pi}_{nm} \right)^2, \tag{77}$$

where $a_0 = 1$, $a_1 = a$, and $a_2 = b$, and q are the coordinates that specify the orientation of the body in space. For $k = 0$, the nonzero coefficients are:

$$I^{(1)}_{0,n,0,0,0} = I^{(1)}_{0,n,0,1,1} = C_n \tag{78}$$

$$I^{(1)}_{0,n,0,2,2} = -2C_n \tag{79}$$

$$I^{(1)}_{0,n,1,0,2} = I^{(1)}_{0,n,2,0,0} = I^{(1)}_{0,n,-1,1,2} = I^{(1)}_{0,n,-2,0,1} = -\sqrt{3} C_n \tag{80}$$

$$I^{(1)}_{0,n,2,1,1} = \sqrt{3} C_n, \tag{81}$$

with $C_1 = 0.1747793752$, $C_2 = -0.0501544874$, and $C_3 = 0.0138166088$, and with $I^{(1)}_{0nmij} = I^{(1)}_{0nmji}$. These have a simple representation $I^{(1)}_{0nmij} = -C_n \partial_i \partial_j \bar{X}_2^m(\omega^a, \omega^b, \omega^c)$.

8 Potential energy

The gravitational potential energy is the integral of Eq. (23) over the mass of the body:

$$V_G(t; q, \Pi; \omega', \dot{\Pi}) = \int_V \rho U_2(\mathbf{x}, \mathbf{R}') dV \tag{82}$$

It is convenient to introduce a number of integrals, and then write the potential energy in terms of them. Let

$$J^{(1)}_{knm} = \frac{1}{V} \int_V \mathbf{u}^k_A \cdot \mathbf{u}^M_{nm} dV, \tag{83}$$

$$J^{(2)}_{nmn'm'} = \frac{1}{V} \int_V \mathbf{u}^M_{nm} \cdot \mathbf{u}^M_{n'm'} dV = \delta_{nn'} \delta_{mm'} \tag{84}$$

$$J^{(3)}_{knmij} = \frac{1}{V} \int_V (\hat{e}_i \cdot \mathbf{u}^k_A)(\hat{e}_j \cdot \mathbf{u}^M_{nm}) dV, \tag{85}$$

$$J^{(4)}_{nmn'm'ij} = \frac{1}{V} \int_V (\hat{e}_i \cdot \mathbf{u}^M_{nm})(\hat{e}_j \cdot \mathbf{u}^M_{n'm'}) dV. \tag{86}$$

In terms of these the gravitational potential energy is

$$V_G(t; q, \Pi; \omega', \dot{\Pi}) \tag{87}$$

$$= -\frac{Gm}{(R')^3} \left(\frac{(1 - 3(\alpha')^2)A + (1 - 3(\beta')^2)B + (1 - 3(\gamma')^2)C}{2} \right) \tag{88}$$

$$+ \frac{GmM}{R'} \left(\frac{R}{R'} \right)^2 \sum_{knm} J^{(1)}_{knm} a_k \Pi_{nm} \tag{89}$$

$$+ \frac{1}{2} \frac{GmM}{R'} \left(\frac{R}{R'} \right)^2 \sum_{nmn'm'} J^{(2)}_{nmn'm'} \Pi_{nm} \Pi_{n'm'} \tag{90}$$

$$- 3 \frac{GmM}{R'} \left(\frac{R}{R'} \right)^2 \sum_{ijknm} J^{(3)}_{knmij} \alpha'_i \alpha'_j a_k \Pi_{nm} \tag{91}$$

$$- \frac{3}{2} \frac{GmM}{R'} \left(\frac{R}{R'} \right)^2 \sum_{ijnmn'm'} J^{(4)}_{nmn'm'ij} \alpha'_i \alpha'_j \Pi_{nm} \Pi_{n'm'}, \tag{92}$$

where $\alpha'_0 = \alpha'$, $\alpha'_1 = \beta'$, $\alpha'_2 = \gamma'$, and $(\alpha', \beta', \gamma')$ are the direction cosines of the perturbing body with respect to the body axes:

$$(\alpha', \beta', \gamma') = (M(q))^{-1} (\alpha, \beta, \gamma), \tag{93}$$

and (α, β, γ) are the direction cosines of the perturbing body with respect to a spatially fixed rectangular basis, and $M(q)$ is the rotation that carries the body in its reference orientation (with principal axes aligned with the spatial axes) to the actual orientation specified by the coordinates q.

The coefficients $J^{(1)}_{0nm}$ are all zero, but there are some nonzero $k = 0$ terms among the $J^{(3)}_{knmij}$. Specifically,

$$J^{(3)}_{0,n,0,0,0} = J^{(3)}_{0,n,0,1,1} = -C_n \tag{94}$$

$$J^{(3)}_{0,n,0,2,2} = 2C_n \tag{95}$$

$$J_{0,n,2,0,0}^{(3)} = J_{0,n,-2,0,1}^{(3)} = J_{0,n,1,0,2}^{(3)} = J_{0,n,-1,1,2}^{(3)} = \sqrt{3}C_n \tag{96}$$

$$J_{0,n,2,1,1}^{(3)} = -\sqrt{3}C_n \tag{97}$$

with C_n as before, and $J_{ij0nm}^{(3)} = J_{ji0nm}^{(3)}$. These have the simple representation $J_{0nmij}^{(3)} = C_n \partial_i \partial_j \bar{X}_2^m(\alpha', \beta', \gamma')$.

Recall that the elastic potential energy, following Eq. (47), is

$$V_E(t; q, \Pi; \omega', \dot{\Pi}) = \frac{1}{2}MR^2 \sum_{nm} \omega_{2n}^2 \Pi_{nm}^2. \tag{98}$$

The total potential energy is $V = V_G + V_E$.

9 Lagrangian and equations of motion

The Lagrangian for the system is the difference of the kinetic energy T and the potential energy V. Note that the kinetic energy does not depend on q and the potential energy does not depend on ω'. Further, the potential energy depends on q only through $(\alpha', \beta', \gamma')$.

It will be convenient to use Euler-like angles to specify the orientation of the body with respect to its reference orientation with the principal axes $(\hat{a}, \hat{b}, \hat{c})$ aligned with the rectangular spatial axes $(\hat{x}, \hat{y}, \hat{z})$. (These are the principal axes of the body without tidal distortion.) We choose the rotation that carries the body to its actual orientation as

$$M(\theta, \phi, \psi) = M_z(\phi)M_x(\theta)M_y(\psi). \tag{99}$$

Note that these are not the usual Euler angles.

The equations of motion for the angular velocities $(\omega^a, \omega^a, \omega^c)$ and the coordinates (θ, ϕ, ψ) are Poincaré's equations (Poincaré 1901). The equations of motion for the velocities $\dot{\Pi}$ and the coordinates Π are the Lagrange equations. In the derivation of the Poincaré equations, we use the vector field basis corresponding to infinitesimal rotations about the body axes $(\hat{a}, \hat{b}, \hat{c})$. In the chosen coordinates, the basis vector fields are

$$e_a = \cos\psi \frac{\partial}{\partial\theta} - \frac{\sin\psi}{\cos\theta} \frac{\partial}{\partial\phi} + \frac{\sin\psi \sin\theta}{\cos\theta} \frac{\partial}{\partial\psi} \tag{100}$$

$$e_b = \frac{\partial}{\partial\psi} \tag{101}$$

$$e_c = \sin\psi \frac{\partial}{\partial\theta} + \frac{\cos\psi}{\cos\theta} \frac{\partial}{\partial\phi} - \frac{\cos\psi \sin\theta}{\cos\theta} \frac{\partial}{\partial\psi} \tag{102}$$

The commutators of these basis fields satisfy

$$[e_i, e_j](f) = \sum_{ijk} c_{ij}^k e_k(f), \tag{103}$$

with structure constants

$$c_{ij}^k = \epsilon_{ijk}, \tag{104}$$

where ϵ_{ijk} is 1 for (i, j, k) equal to a cyclic permutation of $(0, 1, 2)$, -1 for a cyclic permutation of $(2, 1, 0)$, and 0 otherwise. Concretely,

$$[e_a, e_b] = e_c \tag{105}$$

$$[e_b, e_c] = e_a \tag{106}$$

$$[e_c, e_a] = e_b. \tag{107}$$

The Lagrangian depends on the coordinates (θ, ϕ, ψ) only through $(\alpha', \beta', \gamma')$. In the Poincaré equations we need $e_i(\hat{L})$, where $\hat{L}$ is the Lagrangian written in terms of the angular velocities, and the vector field takes the derivative of the coordinate slot of the Lagrangian. We have

$$e_a(\hat{L}) = \gamma' \frac{\partial \hat{L}}{\partial \beta'} - \beta' \frac{\partial \hat{L}}{\partial \gamma'} \tag{108}$$

$$e_b(\hat{L}) = \alpha' \frac{\partial \hat{L}}{\partial \gamma'} - \gamma' \frac{\partial \hat{L}}{\partial \alpha'} \tag{109}$$

$$e_c(\hat{L}) = \beta' \frac{\partial \hat{L}}{\partial \alpha'} - \alpha' \frac{\partial \hat{L}}{\partial \beta'}. \tag{110}$$

The Poincaré equations are, in this case,

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \omega^a}\right) = \gamma' \frac{\partial \hat{L}}{\partial \beta'} - \beta' \frac{\partial \hat{L}}{\partial \gamma'} + \omega^c \frac{\partial \hat{L}}{\partial \omega^b} - \omega^b \frac{\partial \hat{L}}{\partial \omega^c} \tag{111}$$

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \omega^b}\right) = \alpha' \frac{\partial \hat{L}}{\partial \gamma'} - \gamma' \frac{\partial \hat{L}}{\partial \alpha'} + \omega^a \frac{\partial \hat{L}}{\partial \omega^c} - \omega^c \frac{\partial \hat{L}}{\partial \omega^a} \tag{112}$$

$$\frac{d}{dt}\left(\frac{\partial \hat{L}}{\partial \omega^c}\right) = \beta' \frac{\partial \hat{L}}{\partial \alpha'} - \alpha' \frac{\partial \hat{L}}{\partial \beta'} + \omega^b \frac{\partial \hat{L}}{\partial \omega^a} - \omega^a \frac{\partial \hat{L}}{\partial \omega^b} \tag{113}$$

For our Lagrangian, including the $I^{(1)}_{knmij}$ terms in the kinetic energy with $k = 0$, but ignoring the other $I^{(i)}$ contributions, we find

$$\frac{\partial \hat{L}}{\partial \omega^a} = A\omega^a - 2MR^2 \sum_{nm} C_n \partial_0 \bar{X}_2^m(\omega^a, \omega^b, \omega^c) \Pi_{nm} \tag{114}$$

$$\frac{\partial \hat{L}}{\partial \omega^b} = B\omega^b - 2MR^2 \sum_{nm} C_n \partial_1 \bar{X}_2^m(\omega^a, \omega^b, \omega^c) \Pi_{nm} \tag{115}$$

$$\frac{\partial \hat{L}}{\partial \omega^c} = C\omega^c - 2MR^2 \sum_{nm} C_n \partial_2 \bar{X}_2^m(\omega^a, \omega^b, \omega^c) \Pi_{nm} \tag{116}$$

Similarly, keeping the $k = 0$ terms in $J^{(3)}_{knmij}$, but ignoring the other $J^{(i)}$ terms, we find

$$\frac{\partial \hat{L}}{\partial \alpha'} = -\frac{Gm}{(R')^3} 3A\alpha' + MR^2 \sum_{nm} \mathcal{F}_n \Pi_{nm} \partial_0 \bar{X}_2^m(\alpha', \beta', \gamma') \tag{117}$$

$$\frac{\partial \hat{L}}{\partial \beta'} = -\frac{Gm}{(R')^3} 3B\beta' + MR^2 \sum_{nm} \mathcal{F}_n \Pi_{nm} \partial_1 \bar{X}_2^m(\alpha', \beta', \gamma') \tag{118}$$

$$\frac{\partial \hat{L}}{\partial \gamma'} = -\frac{Gm}{(R')^3} 3C\gamma' + MR^2 \sum_{nm} \mathcal{F}_n \Pi_{nm} \partial_2 \bar{X}_2^m(\alpha', \beta', \gamma'), \tag{119}$$

where

$$\mathcal{F}_n = \frac{Gm}{(R')^3}(6C_n) \approx \frac{Gm}{(R')^3}\left(\frac{R\omega_{2n}^2 h_2 g_n}{g}\right), \tag{120}$$

where the last form of $\mathcal{F}_n$ is valid for large rigidity. The Poincaré equations are constructed from these components.

Keeping the same terms as before, the Lagrange equations governing the motion of Π_{nm} are

$$\ddot{\Pi}_{nm} + \omega_{2n}^2 \Pi_{nm} = -2C_n \bar{X}_2^m(\omega^a, \omega^b, \omega^c) + \mathcal{F}_n \bar{X}_2^m(\alpha', \beta', \gamma')). \tag{121}$$

Note that if we ignore the terms involving Π Poincaré's equations become

$$A \frac{d\omega^a}{dt} = (B - C)\omega^b \omega^c - \frac{3Gm}{(R')^3}(B - C)\beta'\gamma' \tag{122}$$

$$B \frac{d\omega^b}{dt} = (C - A)\omega^a \omega^c - \frac{3Gm}{(R')^3}(C - A)\alpha'\gamma' \tag{123}$$

$$C \frac{d\omega^c}{dt} = (A - B)\omega^a \omega^b - \frac{3Gm}{(R')^3}(A - B)\alpha'\beta', \tag{124}$$

which are just Euler's equations for the motion of a rigid body subject to a gravity-gradient torque.

10 Chandler wobble

Elasticity affects the period of the Eulerian wobble of the Earth (Chandler 1891; Newcomb 1882). As an illustration of the formalism we have developed we will calculate the elastic correction to the frequency of the Eulerian wobble. We assume the body is rotating very nearly with its spin axis aligned with the $\hat{c}$ principal axis, and that the body is axisymmetric $A = B$. We assume there is no external perturbing body. For simplicity we take into account only the gravest elastic modes (those with $n = 1$). We assume here that the elastic modes are in equilibrium, so, from Eq. (121), we have

$$\Pi_{1m} = -2C_1 \bar{X}_2^m(\omega^a, \omega^b, \omega^c)/\omega_{21}^2. \tag{125}$$

We will assume ω^a and ω^b are much smaller than ω^c; we will systematically ignore terms that are second and higher order in ω^a and ω^b. With this assumption

$$\Pi_{1,0} = -2C_1 \frac{(\omega^c)^2}{\omega_{21}^2} \tag{126}$$

$$\Pi_{1,1} = -2\sqrt{3}C_1 \frac{\omega^a \omega^c}{\omega_{21}^2} \tag{127}$$

$$\Pi_{1,-1} = -2\sqrt{3}C_1 \frac{\omega^b \omega^c}{\omega_{21}^2} \tag{128}$$

$$\Pi_{1,2} = 0 \tag{129}$$

$$\Pi_{1,-2} = 0. \tag{130}$$

We have, in this case,

$$\frac{\partial \hat{L}}{\partial \omega^a} = A\omega^a + 2C_1 MR^2 \Pi_{1,0}\omega^a - 2\sqrt{3}C_1 MR^2 \Pi_{1,1}\omega^c \tag{131}$$

$$\frac{\partial \hat{L}}{\partial \omega^b} = B\omega^b + 2C_1 MR^2 \Pi_{1,0}\omega^b - 2\sqrt{3}C_1 MR^2 \Pi_{1,-1}\omega^c \tag{132}$$

$$\frac{\partial \hat{L}}{\partial \omega^c} = C\omega^c - 4C_1 MR^2 \Pi_{1,0}\omega^c. \tag{133}$$

The Poincaré equations, with these assumptions, show that ω^c is constant, so $\Pi_{1,0}$ is also constant. We define

$$A' = A + 2C_1 MR^2 \Pi_{10} = A - 4C_1^2 MR^2 \left(\frac{\omega^c}{\omega_{21}}\right)^2 \tag{134}$$

$$B' = B + 2C_1 MR^2 \Pi_{10} = B - 4C_1^2 MR^2 \left(\frac{\omega^c}{\omega_{21}}\right)^2 \tag{135}$$

$$C' = C - 4C_1 MR^2 \Pi_{10} = C + 8C_1^2 MR^2 \left(\frac{\omega^c}{\omega_{21}}\right)^2, \tag{136}$$

and assume that (A', B', C') are the observed principal moments of inertia. The Poincaré equations are then

$$A'\dot{\omega}^a \left(1 + 12C_1^2 \frac{MR^2}{A'}\left(\frac{\omega^c}{\omega_{21}}\right)^2\right) = \omega^b \omega^c (B' - C') + 12C_1^2 \left(\frac{\omega^c}{\omega_{21}}\right)^2 MR^2 \omega^b \omega^c \tag{137}$$

$$B'\dot{\omega}^b \left(1 + 12C_1^2 \frac{MR^2}{B}\left(\frac{\omega^c}{\omega_{21}}\right)^2\right) = \omega^a \omega^c (C' - A') - 12C_1^2 \left(\frac{\omega^c}{\omega_{21}}\right)^2 MR^2 \omega^a \omega^c \tag{138}$$

Taking the time derivative of the first equation and using the second, we find

$$\ddot{\omega}^a = -\omega_C^2 \omega^a, \tag{139}$$

where the Chandler frequency is

$$\omega_C = \omega^c \left(\frac{C' - A'}{A'} - 12C_1^2 \frac{MR^2}{A}\left(\frac{\omega^c}{\omega_{21}}\right)^2\right), \tag{140}$$

and we have ignored a quantity of second order. We see that the Eulerian frequency, $\omega^c(C' - A')/A'$, is reduced by elasticity. As so far developed, our theory does not apply to a body such as the Earth where the body has significant radial variation in density. Nevertheless, we can see what effective rigidity a homogeneous Earth must have to get the observed Chandler period of 434 days. Using a density of $5500\,\mathrm{kg/m^3}$ and a radius of $6{,}371{,}000\,\mathrm{m}$, $C'/(MR^2) = 0.3307$, and $A'/(MR^2) = 0.3296$, we find $\mu = 1.8 \times 10^{11}\,\mathrm{N/m^2}$, which is actually comparable to modern estimates of the rigidity of the Earth at a depth of about $1000\,\mathrm{km}$ (Stacey 1992).

11 Dissipation

The Q of an oscillator is defined as

$$\frac{1}{Q} = \frac{1}{2\pi E^\star} \oint \frac{dE}{dt}\,dt, \tag{141}$$

where dE/dt is the rate at which work is done on the oscillator, $E^\star$ is the energy stored in the oscillator, and the integral is over one cycle of the oscillation. Consider the damped driven oscillator

$$m(\ddot{x} + d\dot{x} + \omega_0^2 x) = A\cos(\omega t). \tag{142}$$

The forced response is

$$x(t) = B\cos(\omega t - \delta), \tag{143}$$

where

$$B^2 = \left(\frac{A}{m}\right)^2 \frac{1}{(\omega_0^2 - \omega^2)^2 + (\omega d)^2}. \tag{144}$$

Let

$$\delta_0 = \arctan(\omega d, \omega_0^2 - \omega^2), \tag{145}$$

then for $\omega < \omega_0$ the phase shift $\delta = \delta_0$ is positive, meaning that the response of the oscillator lags the forcing, and B is positive. For $\omega > \omega_0$, the phase shift δ_0 is greater than $\pi/2$. We could take $\delta = \delta_0$ for all ω and always keep B positive. Alternatively, we can bring δ into the range of $-\pi/2$ to $\pi/2$ (the range of the one argument arctan) by subtracting π from δ_0 making $\delta = \delta_0 - \pi$ negative. Then we flip the sign of B to be negative, so that the solution remains valid. So for $\omega < \omega_0$, both δ and B are positive, and for $\omega > \omega_0$ both δ and B are negative. This reflects more clearly that the response is out of phase with the forcing.

The energy dissipated per cycle is

$$\Delta E = \pi A B \sin \delta, \tag{146}$$

which is positive. For now, we take the peak energy stored in the oscillator to be

$$E^\star = m\omega_0^2 B^2/2, \tag{147}$$

and return to this definition later. The Q is then given by

$$\frac{1}{Q} = \frac{((\omega_0^2 - \omega^2)^2 + (\omega d)^2)^{1/2}}{\omega_0^2} \sin \delta \tag{148}$$

$$\approx \frac{((\omega_0^2 - \omega^2)^2)^{1/2}}{\omega_0^2} \sin \delta \tag{149}$$

$$\approx \sin \delta, \tag{150}$$

where the first approximation is for small dissipation (small d), and the second approximation additionally assumes small forcing frequency ($\omega \ll \omega_0$). In the same limit

$$\tan \delta \approx \omega d/\omega_0^2. \tag{151}$$

So for large Q

$$\frac{1}{Q} \approx \frac{\omega d}{\omega_0^2}. \tag{152}$$

Note that in this model, with these definitions, and in these limits the Q is inversely proportional to ω.

The definition of the energy stored is somewhat problematic. In Eq. (147) we took the energy stored to be the maximum potential energy of the spring, the potential energy of the spring at its maximum extension. Goldreich (1963) took a different definition: he defined $E^\star$ to be the integral over the first quarter cycle of the rate at which work is done on the oscillator:

$$E^\star = \int_0^{T/4} F v \, dt \tag{153}$$

$$= -\int_0^{T/4} (A\cos(\omega t)\omega B \sin(\omega t - \delta)) \, dt \tag{154}$$

$$= AB\omega \left(\frac{\cos(2\omega t - \delta)}{4\omega} - \frac{t \sin \delta}{2} \right) \Bigg|_0^{T/4} \tag{155}$$

$$= \frac{AB}{2} \left(\cos(\delta) - \frac{\pi}{2} \sin \delta \right) \tag{156}$$

Ignoring the second term term in $E^\star$ for small δ, we find that

$$\frac{1}{Q} = \tan \delta \tag{157}$$

More exactly, there is a second term, which Goldreich ignores as an approximation. Peale (2011, personal communication) argues that the second term should be excluded in general because the energy stored in the oscillator should be the integral of the conservative rate of work and the second term corresponds to dissipative work. Thus Eq. (157) should be regarded as exact. We may compare this to the result we derived above, Eq. (149), in which $1/Q$ is approximately $\sin \delta$ and then only for small d and $\omega \ll \omega_0$. For this model, to distinguish these is splitting hairs because the phase lags are not known well enough to distinguish between $\tan \delta$ and $\sin \delta$. But in other models of dissipation, different definitions of the energy stored lead to more qualitatively different results.

We incorporate tidal dissipation into our modal model by adding an ad hoc dissipative term to the modal equations, Eq. (121),

$$\ddot{\Pi}_{nm} + d_n \dot{\Pi}_{nm} + \omega_{2n}^2 \Pi_{nm} = -2 C_n \bar{X}_2^m(\omega^a, \omega^b, \omega^c) + \mathcal{F}_n \bar{X}_2^m(\alpha', \beta', \gamma'). \tag{158}$$

We assume d_n is independent of m. We have

$$\frac{1}{Q_n} \approx \frac{\omega d_n}{\omega_{2n}^2}, \tag{159}$$

where ω is the forcing frequency, assumed to be much less than the modal frequency.

The fact that this model gives phase lags proportional to frequency makes it equivalent to popularly used sets of tidal evolution equations (Mignard 1981; Hut 1981). The disadvantage is that real materials and planets do not behave this way. The frequency dependence of the phase lags for the Moon are different (Williams 2008). In a subsequent paper we will incorporate different rheologies into our model.

12 Wobble damping

Here we consider the decay of the Eulerian wobble due to dissipation in the elastic modes. We again assume that the spin axis is nearly aligned with the $\hat{c}$ axis. The damped modal equations (keeping only $n = 1$ modes) are

$$\ddot{\Pi}_{1,0} + d_1 \dot{\Pi}_{1,0} + \omega_{21}^2 \Pi_{1,0} = -2 C_1 (\omega^c)^2 \tag{160}$$

$$\ddot{\Pi}_{1,1} + d_1 \dot{\Pi}_{1,1} + \omega_{21}^2 \Pi_{1,1} = -2\sqrt{3} C_1 \omega^c \omega^a \tag{161}$$

$$\ddot{\Pi}_{1,-1} + d_1 \dot{\Pi}_{1,-1} + \omega_{21}^2 \Pi_{1,-1} = -2\sqrt{3} C_1 \omega^c \omega^b. \tag{162}$$

As ω^c is constant, we can assume

$$\Pi_{1,0} = -2 C_1 (\omega^c / \omega_{21})^2, \tag{163}$$

is constant. To construct the Poincaré equations we need

$$\frac{\partial \hat{L}}{\partial \omega^a} = A\omega^a + 2C_1 MR^2 \Pi_{1,0}\omega^a - 2\sqrt{3}C_1 MR^2 \Pi_{1,1}\omega^c \tag{164}$$

$$\frac{\partial \hat{L}}{\partial \omega^b} = B\omega^b + 2C_1 MR^2 \Pi_{1,0}\omega^b - 2\sqrt{3}C_1 MR^2 \Pi_{1,-1}\omega^c \tag{165}$$

ignoring second order quantities. Define (A', B', C') as before, and assume these are the observed values of the principal moments. Then the left hand sides of the Poincaré equations are

$$\frac{d}{dt}\frac{\partial \hat{L}}{\partial \omega^a} = A'\dot{\omega}^a - 2\sqrt{3}C_1 MR^2 \omega^c \dot{\Pi}_{1,1} \tag{166}$$

$$\frac{d}{dt}\frac{\partial \hat{L}}{\partial \omega^a} = B'\dot{\omega}^b - 2\sqrt{3}C_1 MR^2 \omega^c \dot{\Pi}_{1,-1} \tag{167}$$

and the right hand sides are

$$\omega^c \frac{\partial \hat{L}}{\partial \omega^b} - \omega^b \frac{\partial \hat{L}}{\partial \omega^c} = \omega^c \omega^b (B' - C') - 2\sqrt{3}C_1 MR^2 (\omega^c)^2 \Pi_{1,-1} \tag{168}$$

$$\omega^a \frac{\partial \hat{L}}{\partial \omega^c} - \omega^c \frac{\partial \hat{L}}{\partial \omega^a} = \omega^c \omega^a (C' - A') + 2\sqrt{3}C_1 MR^2 (\omega^c)^2 \Pi_{1,1}. \tag{169}$$

Collecting the equations, the wobble damping is governed by the constant coefficient linear differential equations:

$$0 = \ddot{\Pi}_{1,1} + d_1 \dot{\Pi}_{1,1} + \omega_{21}^2 \Pi_{1,1} + \xi \omega^c \omega^a \tag{170}$$

$$0 = \ddot{\Pi}_{1,-1} + d_1 \dot{\Pi}_{1,-1} + \omega_{21}^2 \Pi_{1,-1} + \xi \omega^c \omega^b \tag{171}$$

$$0 = A'\dot{\omega}^a - \xi \omega^c MR^2 \dot{\Pi}_{1,1} - \omega^c \omega^b (B' - C') + \xi \omega^c MR^2 \omega^c \Pi_{1,-1} \tag{172}$$

$$0 = B'\dot{\omega}^b - \xi \omega^c MR^2 \dot{\Pi}_{1,-1} - \omega^c \omega^a (C' - A') - \xi \omega^c MR^2 \omega^c \Pi_{1,1} \tag{173}$$

where $\xi = 2\sqrt{3}C_1$. We find the solution decays exponentially, proportional to $e^{-t/\tau}$, with

$$\frac{1}{\tau} = \xi^2 \left(\frac{\omega^c}{\omega_{21}}\right)^4 \left(\frac{MR^2}{A'}\right) \left(\frac{\omega_C}{\omega^c}\right) d_1. \tag{174}$$

Using Eq. (152), we define the effective Q for an oscillation at the Chandler frequency to satisfy

$$\frac{1}{Q_C} = \frac{\omega_C d_1}{\omega_{21}^2}, \tag{175}$$

then

$$\frac{1}{\tau} = \xi^2 \frac{(\omega^c)^3}{(\omega_{21})^2} \left(\frac{MR^2}{A'}\right) \left(\frac{1}{Q_C}\right). \tag{176}$$

The wobble damping timescale is approximately

$$\tau \approx 19.38 \frac{A}{MR^2} \frac{\mu Q_C}{\rho R^2 (\omega^c)^3}. \tag{177}$$

This agrees with Peale (1973), though the method of calculation is very different.

13 Tidal friction

Consider a perturbing body in orbit about a dissipative elastic body. We assume the spin axis of the body is perpendicular to the orbit plane ($\gamma' = 0$) and that there is no wobble ($\omega^a = \omega^b = 0$). We will use the same coordinates (θ, ϕ, ψ) to specify the orientation as before. Let n be the mean motion of the perturbing body, with true longitude λ. We assume that $A' = B'$. We will consider only the gravest elastic modes (those with $n = 1$). With these assumptions the equations of motion are

$$C'\dot{\omega}^c = \sqrt{3}MR^2\mathcal{F}_1\left(\Pi_{1,2}2\alpha'\beta' - \Pi_{1,-2}\left((\alpha')^2 - (\beta')^2\right)\right) \tag{178}$$

$$\dot{\omega}^a = \dot{\omega}^b = 0 \tag{179}$$

with

$$\ddot{\Pi}_{1,2} + d_1\dot{\Pi}_{1,2} + \omega_{21}^2\Pi_{1,2} = \mathcal{F}_1\frac{\sqrt{3}}{2}\left((\alpha')^2 - (\beta')^2\right) \tag{180}$$

$$\ddot{\Pi}_{1,-2} + d_1\dot{\Pi}_{1,-2} + \omega_{21}^2\Pi_{1,-2} = \mathcal{F}_1\frac{\sqrt{3}}{2}2\alpha'\beta' \tag{181}$$

$$\Pi_{1,1} = \Pi_{1,-1} = 0, \tag{182}$$

and with

$$\Pi_{1,0} \approx -(2C_1(\omega^c)^2 + \mathcal{F}_1/2)/\omega_{21}^2. \tag{183}$$

The average polar moment is

$$C' = C + 8C_1^2MR^2\left(\frac{\omega^c}{\omega_{21}}\right)^2 + 2C_1MR^2\frac{\mathcal{F}_1}{\omega_{21}^2}. \tag{184}$$

For this geometry $\alpha' = \cos(\lambda - \phi)$ and $\beta' = \sin(\lambda - \phi)$, with $\omega^c = \dot{\phi}$. Define $\phi' = \phi - \lambda$, then the equations of motion are

$$C'\dot{\omega}^c = \sqrt{3}MR^2\mathcal{F}_1\left(-\Pi_{1,2}\sin(2\phi') - \Pi_{1,-2}\cos(2\phi')\right) \tag{185}$$

$$\ddot{\Pi}_{1,2} + d_1\dot{\Pi}_{1,2} + \omega_{21}^2\Pi_{1,2} = \mathcal{F}_1\frac{\sqrt{3}}{2}\cos(2\phi') \tag{186}$$

$$\ddot{\Pi}_{1,-2} + d_1\dot{\Pi}_{1,-2} + \omega_{21}^2\Pi_{1,-2} = -\mathcal{F}_1\frac{\sqrt{3}}{2}\sin(2\phi') \tag{187}$$

Let's assume that the rotation is not near synchronous, and that ϕ' moves approximately uniformly

$$\phi' \approx (\omega^c - n)t. \tag{188}$$

(We are taking the orbital eccentricity to be zero.) Then the Π equations are periodically forced damped harmonic oscillators, with solutions (assuming $\omega_{21} \gg |\omega^c - n|$)

$$\Pi_{1,2} = \frac{\sqrt{3}}{2}\frac{\mathcal{F}_1}{\omega_{21}^2}\cos(2(\omega^c - n)t - \delta_{2\omega^c - 2n}) \tag{189}$$

$$\Pi_{1,-2} = -\frac{\sqrt{3}}{2}\frac{\mathcal{F}_1}{\omega_{21}^2}\sin(2(\omega^c - n)t - \delta_{2\omega^c - 2n}) \tag{190}$$

where

$$\tan\delta_{2\omega^c - 2n} \approx \frac{2(\omega^c - n)d_1}{\omega_{21}^2} = \frac{1}{Q_T}, \tag{191}$$

defining Q_T as the effective Q at the frequency $2(\omega^c - n)$. Substituting these into the equation for $\dot{\omega}^c$ we find

$$\dot{\omega}^c = -\frac{3}{2} \frac{MR^2}{C'} \frac{(\mathcal{F}_1)^2}{\omega_{21}^2} \sin \delta_{2\omega^c - 2n}. \tag{192}$$

This gives the rate of deceleration of rotation for a perturber of mass m in a circular orbit. It can apply to either the case of the deceleration of a planet by a satellite or of the rotation of a satellite by the primary. This expression is easily generalized using Eqs. (144–148).

To reduce this expression to the usual expression we have to make an additional assumption. For one of the factors of $\mathcal{F}_1$, we use the approximate form for large rigidity in Eq. (120). We also use $k_2 = (3/5)h_2$. We find then

$$\dot{\omega}^c = -\frac{3}{2} \frac{k_2}{Q_T} \left(\frac{R}{R'}\right)^3 \left(\frac{Gm}{a^3}\right) \left(\frac{m}{M}\right) \frac{MR^2}{C'} \chi, \tag{193}$$

where

$$\chi = 10 C_1 g_1 \approx 0.98. \tag{194}$$

In each case m is the tidal perturber, and M is the perturbed solid body. Except for the χ factor Eq. (193) gives the expression in Goldreich and Peale (1966). Note that the expression given by Peale (1973) is too big by a factor of 2, as mentioned by Dobrovolskis (2007). Also note if we include more modes then the factor of χ is closer to 1: $\sum_{i=1}^{\infty} 10 C_i g_i = 1$.

This formula can be applied to get the tidal decay of rotation for a non-synchronous satellite, but also for the tidal evolution of semimajor axis. Assuming the angular momentum of the system is conserved, the angular momentum lost by the rotating body goes into the orbit. We find, using Eq. (193), that the rate of tidal evolution of the semimajor axis a of a circular orbit is

$$\frac{1}{a} \frac{da}{dt} = 3n \frac{k_2}{Q_T} \frac{R^5}{a^5} \frac{m}{M}, \tag{195}$$

which agrees with Peale (1986).

14 Tidal friction with eccentricity

Consider a perturbing body in an elliptic orbit about a dissipative elastic body. To start, we will assume the spin axis of the body is perpendicular to the orbit plane ($\gamma' = 0$) and that there is no wobble ($\omega^a = \omega^b = 0$). Let n be the mean motion of the perturbing body, a the semimajor axis, e the orbital eccentricity, ϖ the longitude of pericenter, and λ the true longitude. Following, Eqs. (185–187), the equations of motion are

$$C' \dot{\omega}^c = \sqrt{3} M R^2 \mathcal{F}_1 \left(-\Pi_{1,2} \sin(2(\phi - \lambda)) - \Pi_{1,-2} \cos(2(\phi - \lambda))\right) \tag{196}$$

$$\ddot{\Pi}_{1,2} + d_1 \dot{\Pi}_{1,2} + \omega_{21}^2 \Pi_{1,2} = \mathcal{F}_1 \frac{\sqrt{3}}{2} \cos(2(\phi - \lambda)) \tag{197}$$

$$\ddot{\Pi}_{1,-2} + d_1 \dot{\Pi}_{1,-2} + \omega_{21}^2 \Pi_{1,-2} = -\mathcal{F}_1 \frac{\sqrt{3}}{2} \sin(2(\phi - \lambda)), \tag{198}$$

$$\ddot{\Pi}_{1,0} + d_1 \dot{\Pi}_{1,0} + \omega_{21}^2 \Pi_{1,0} = -(2C_1(\omega^c)^2 + \mathcal{F}_1/2), \tag{199}$$

where both $\mathcal{F}_1$ and λ vary nonuniformly because of the Keplerian orbital motion. Recall that $\mathcal{F}_1 \propto (a/R'(t))^3$ (see Eq. 120).

To solve these equations we Fourier expand the periodic forcing. We use

$$\left(\frac{a}{R'(t)}\right)^3 \cos(kf(t)) = \sum_{j=-\infty}^{\infty} X_j^{-3,k}(e) \cos(jnt), \tag{200}$$

and

$$\left(\frac{a}{R'(t)}\right)^3 \sin(kf(t)) = \sum_{j=-\infty}^{\infty} X_j^{-3,k}(e) \sin(jnt), \tag{201}$$

where f is the true anomaly and $X_j^{-3,k}(e)$ are Hansen functions.

Substituting these into the Π equations of motion gives

$$\ddot{\Pi}_{1,2} + d_1 \dot{\Pi}_{1,2} + \omega_{21}^2 \Pi_{1,2} = \mathcal{G}_1 \frac{\sqrt{3}}{2} \sum_{j=-\infty}^{\infty} X_j^{-3,2}(e) \cos(2\phi - 2\varpi - jnt) \tag{202}$$

$$\ddot{\Pi}_{1,-2} + d_1 \dot{\Pi}_{1,-2} + \omega_{21}^2 \Pi_{1,-2} = -\mathcal{G}_1 \frac{\sqrt{3}}{2} \sum_{j=-\infty}^{\infty} X_j^{-3,2}(e) \sin(2\phi - 2\varpi - jnt) \tag{203}$$

$$\ddot{\Pi}_{1,0} + d_1 \dot{\Pi}_{1,0} + \omega_{21}^2 \Pi_{1,0} = -2C_1(\omega^c)^2 - \mathcal{G}_1 \frac{1}{2} \sum_{j=-\infty}^{\infty} X_j^{-3,0}(e) \cos(jnt), \tag{204}$$

where $\phi = \phi_0 + \omega^c t$ (with ϕ_0 giving the orientation of the planet at time $t = 0$) and

$$\mathcal{G}_1 \left(\frac{a}{R'(t)}\right)^3 = \mathcal{F}_1. \tag{205}$$

These linear equations have the approximate solutions

$$\Pi_{1,2} = \frac{\sqrt{3}}{2} \frac{\mathcal{G}_1}{\omega_{21}^2} \sum_{j=-\infty}^{\infty} X_j^{-3,2}(e) \cos(2\phi - 2\varpi - jnt - \delta_{2\omega^c - jn}) \tag{206}$$

$$\Pi_{1,-2} = -\frac{\sqrt{3}}{2} \frac{\mathcal{G}_1}{\omega_{21}^2} \sum_{j=-\infty}^{\infty} X_j^{-3,2}(e) \sin(2\phi - 2\varpi - jnt - \delta_{2\omega^c - jn}) \tag{207}$$

$$\Pi_{1,0} = -2C_1 \frac{(w^c)^2}{\omega_{21}^2} - \frac{1}{2} \frac{\mathcal{G}_1}{\omega_{21}^2} \sum_{j=-\infty}^{\infty} X_j^{-3,0}(e) \cos(jnt - \delta_{jn}) \tag{208}$$

where

$$\tan \delta_{2\omega^c - jn} \approx \frac{(2\omega^c - jn)d_1}{\omega_{21}^2}, \tag{209}$$

$$\tan \delta_{jn} \approx \frac{jnd_1}{\omega_{21}^2}. \tag{210}$$

In writing these expressions we have used the approximation that $|2\omega^c - jn| << \omega_{21}$. These expressions are easily generalized.

Substituting these expressions into the equation for $\dot{\omega}^c$ and averaging over the orbital period, we find

$$\dot{\omega}^c = -\frac{3}{2} \frac{MR^2}{C'} \frac{(\mathcal{G}_1)^2}{\omega_{21}^2} \sum_{j=-\infty}^{\infty} \left(X_j^{-3,2}(e)\right)^2 \sin \delta_{2\omega^c - jn}, \tag{211}$$

where we have ignored the small time variation in C' which exists because $\mathcal{F}_1$ (see Eq. 184) is now time dependent. Note that $X_0^{-3,2}(e) = 0$. This expression looks more familiar if we assume large rigidity:

$$\dot{\omega}^c = -\frac{3}{2}k_2 \left(\frac{R}{R'}\right)^3 \left(\frac{Gm}{a^3}\right)\left(\frac{m}{M}\right)\frac{MR^2}{C'}\sum_{j=-\infty}^{\infty}\left(X_j^{-3,2}(e)\right)^2 \sin\delta_{2\omega^c-jn}, \qquad (212)$$

where we have ignored the factor of χ.

Next we compute the total energy dissipation. For the simple driven oscillator the rate of dissipation is given by Eq. (146). Generalizing this to our case, where there are many forcing terms, and integrating over an orbit period, we find the average rate of energy dissipation to be

$$\frac{dE}{dt} = \frac{3}{4}\frac{MR^2\mathcal{G}_1^2}{\omega_{21}^2}\sum_{j=-\infty}^{\infty}\left(X_j^{-3,2}(e)\right)^2 (2\omega^c - jn)\sin\delta_{2\omega^c-jn} \qquad (213)$$

$$+\frac{1}{4}\frac{MR^2\mathcal{G}_1^2}{\omega_{21}^2}\sum_{j=-\infty}^{\infty}\left(X_j^{-3,0}(e)\right)^2 (jn)\sin\delta_{jn}. \qquad (214)$$

For the special case of synchronous rotation we set $\omega^c = n$.

To derive the usual expression for tidal heating in synchronous rotation at low eccentricity, we select the lowest order terms in eccentricity:

$$\frac{dE}{dt} = \frac{21}{2}\frac{MR^2\mathcal{G}_1^2}{\omega_{21}^2}e^2\sin\delta_n, \qquad (215)$$

where $\delta_n = -\delta_{-n} = \delta_{2\omega^c-n} = -\delta_{2\omega^c-3n}$. To get the familiar formula, we make a large μ approximation (as before) and find

$$\frac{dE}{dt} = \frac{21}{2}\frac{k_2}{Q}\frac{Gm^2nR^5}{a^6}e^2, \qquad (216)$$

where we have again ignored a factor of $\chi = 0.98$ and approximated $\sin\delta_n$ by $1/Q$. This agrees with Peale and Cassen (1978) and Peale et al. (1979).

For the case of phase lags proportional to frequency we use $\delta_{2\omega^c-jn} = (2-j)\delta_n$ and $\delta_{jn} = j\delta_n$. Making the large μ approximation again, we derive

$$\frac{dE}{dt} = \frac{21}{2}\frac{k_2}{Q}\frac{Gm^2nR^5}{a^6}e^2\eta(e), \qquad (217)$$

where the tidal heating enhancement factor is

$$\eta(e) = 1 + 18e^2 + \frac{3329}{28}e^4 + \frac{55551}{112}e^6 + \cdots. \qquad (218)$$

This agrees with Peale and Cassen (1978) and Wisdom (2008). Wisdom (2008) gave an expression valid at arbitrary eccentricity:

$$\zeta(e) = e^2\eta(e) = \frac{2}{7}\frac{f_0(e)}{\beta^{15}} - \frac{4}{7}\frac{f_1(e)}{\beta^{12}} + \frac{2}{7}\frac{f_2(e)}{\beta^9}, \qquad (219)$$

where

$$f_0(e) = 1 + \frac{31}{2}e^2 + \frac{255}{8}e^4 + \frac{185}{16}e^6 + \frac{25}{64}e^8, \tag{220}$$

$$f_1(e) = 1 + \frac{15}{2}e^2 + \frac{45}{8}e^4 + \frac{5}{16}e^6, \tag{221}$$

$$f_2(e) = 1 + 3e^2 + \frac{3}{8}e^4, \tag{222}$$

with $\beta = (1 - e^2)^{1/2}$.

14.1 Orbital evolution not in spin-orbit resonance

Next, we can use conservation of energy and angular momentum with the above results to compute the rates of change of semimajor axis and eccentricity for the case in which there is no spin-orbit resonance. Based on conservation of energy,

$$\frac{da}{dt} = -\frac{2a^2}{GMm}\left(C\omega^c\dot{\omega}^c + \frac{dE}{dt}\right). \tag{223}$$

Substituting in Eqs. (211) and (214) yields

$$\frac{1}{a}\frac{da}{dt} = \frac{\kappa}{2}\sum_{j=-\infty}^{\infty} j\left(3(X_j^{-3,2}(e))^2 \sin\delta_{2\omega^c-jn} - (X_j^{-3,0}(e))^2 \sin\delta_{jn}\right), \tag{224}$$

where

$$\kappa = \frac{a}{GMm}MR^2\frac{\mathcal{G}_1^2}{\omega_{21}^2} \approx \frac{k_2 n R^5}{a^5}\frac{m}{M}, \tag{225}$$

where the approximation is for large rigidity. Conservation of angular momentum gives

$$\frac{1}{e}\frac{de}{dt} = \frac{\kappa}{4}\frac{\sqrt{1-e^2}}{e^2}\left(-\sqrt{1-e^2}\sum_{j=-\infty}^{\infty}(X_j^{-3,0}(e))^2 j \sin\delta_{jn}\right. \tag{226}$$

$$\left. + 3\sum_{j=-\infty}^{\infty}\left(j\sqrt{1-e^2}-2\right)(X_j^{-3,2}(e))^2 \sin\delta_{2\omega^c-jn}\right). \tag{227}$$

Note that the leading term in $(1/e)\,de/dt$ as a polynomial in e is proportional to a constant— the $1/e^2$ factor is cancelled by a leading factor of e^2 in the subsequent expression.

If we keep only the lowest order terms in eccentricity, the expressions reduce to

$$\frac{1}{a}\frac{da}{dt} = 3\kappa \sin\delta_{2\omega^c-2n} \tag{228}$$

and

$$\frac{1}{e}\frac{de}{dt} = -\frac{3\kappa}{4}\left(\frac{3}{2}\sin\delta_n + \frac{1}{4}\sin\delta_{2\omega^c-n} + \sin\delta_{2\omega^c-2n} - \frac{49}{4}\sin\delta_{2\omega^c-3n}\right), \tag{229}$$

which confirms Equation 7 of Goldreich (1963). (Goldreich's ϵ_3 is actually δ_n, which is associated with frequency n, not $\frac{3}{2}n$, as stated.)

For Mignard tides, where phase lags are proportional to frequency,

$$\frac{1}{a}\frac{da}{dt} = 6\frac{\kappa}{Q}\left[\left(\frac{\omega^c}{n} - 1\right) + e^2\left(\frac{27}{2}\frac{\omega^c}{n} - 23\right) + e^4\left(\frac{573}{8}\frac{\omega^c}{n} - 180\right)\right.$$
$$\left. + e^6\left(\frac{3961}{16}\frac{\omega^c}{n} - \frac{6765}{8}\right) + \cdots\right] \tag{230}$$

$$\frac{1}{e}\frac{de}{dt} = \frac{33}{2}\frac{\kappa}{Q}\left[\left(\frac{\omega^c}{n} - \frac{18}{11}\right) + e^2\left(\frac{13}{2}\frac{\omega^c}{n} - \frac{369}{22}\right)\right.$$
$$\left. + e^4\left(\frac{181}{8}\frac{\omega^c}{n} - \frac{3645}{44}\right) + \cdots\right]. \tag{231}$$

14.2 Orbital evolution in spin-orbit resonance

Similar calculations yield the rates of change of semimajor axis and eccentricity in spin-orbit resonance, where $\omega^c = kn$ and therefore

$$\dot{\omega}^c = -\frac{3}{2}k\frac{n}{a}\frac{da}{dt}. \tag{232}$$

For the case of a tide-raising perturber orbiting around an extended body,

$$\frac{1}{a}\frac{da}{dt} = \frac{2a}{GMm}\left(3k^2\Gamma_m - 1\right)^{-1}\frac{dE}{dt}, \tag{233}$$

with $\Gamma_m = C/(ma^2)$. Substituting in Eq. (214) yields

$$\frac{1}{a}\frac{da}{dt} = \frac{\kappa}{2}\left(3k^2\Gamma_m - 1\right)^{-1}\Upsilon, \tag{234}$$

where

$$\Upsilon = \sum_{j=-\infty}^{\infty}\left(3(X_j^{-3,2}(e))^2(2k-j)\sin\delta_{2\omega^c-jn} + (X_j^{-3,0}(e))^2 j\sin\delta_{jn}\right). \tag{235}$$

Conservation of angular momentum gives

$$\frac{1}{e}\frac{de}{dt} = -\frac{\kappa}{4}\frac{\sqrt{1-e^2}}{e^2}\left(\frac{\sqrt{1-e^2} - 3k\Gamma_m}{1 - 3k^2\Gamma_m}\right)\Upsilon \tag{236}$$

Note that the first parenthesized factor in Eq. (236) is approximately one for small eccentricity in synchronous rotation, in both the limit that $\Gamma_m = C/(ma^2)$ is large and small compared to one. Both these limits occur in the Solar System: for the satellites of the giant planets $\Gamma_m \gg 1$, whereas for the Moon $\Gamma_m \ll 1$.

For the case of an extended body orbiting around a perturbing central mass,

$$\frac{1}{a}\frac{da}{dt} = \frac{\kappa}{2}\left(3k^2\Gamma_M - 1\right)^{-1}\Upsilon, \tag{237}$$

with $\Gamma_M = C/(Ma^2)$, and

$$\frac{1}{e}\frac{de}{dt} = -\frac{\kappa}{4}\frac{\sqrt{1-e^2}}{e^2}\left(\frac{\sqrt{1-e^2} - 3k\Gamma_M}{1 - 3k^2\Gamma_M}\right)\Upsilon \tag{238}$$

Note that $\Gamma_M = C/(Ma^2)$ is typically small compared to one.

For Mignard tides we can expand Υ as a power series in e

$$\Upsilon = 12(k-1)^2 + e^2(90k^2 - 324k + 276) + e^4(315k^2 - 1719k + 2160) \quad (239)$$

$$+\frac{1}{2}e^6(1575k^2 - 11883k + 20295) + \cdots \quad (240)$$

Notice that for synchronous rotation the leading term in Υ is order e^2, but for other spin-orbit commensurabilities the leading term is a constant. Thus a satellite would not be expected to be in nonsynchronous spin-orbit resonance at low eccentricity. The eccentricity would decay rapidly to zero in finite time and then the resonance lock would be lost.

If we specialize to synchronous rotation (k=1) and keep only the lowest order terms in eccentricity, the expressions reduce to

$$\frac{1}{a}\frac{da}{dt} = \frac{3\kappa}{4}e^2 (3\Gamma_m - 1)^{-1}\left(3\sin\delta_n + \frac{1}{2}\sin\delta_{2\omega^c - n} - \frac{49}{2}\sin\delta_{2\omega^c - 3n}\right) \quad (241)$$

for a perturber with mass m orbiting around a synchronous extended body with mass M and

$$\frac{1}{a}\frac{da}{dt} = \frac{3\kappa}{4}e^2 (3\Gamma_M - 1)^{-1}\left(3\sin\delta_n + \frac{1}{2}\sin\delta_{2\omega^c - n} - \frac{49}{2}\sin\delta_{2\omega^c - 3n}\right) \quad (242)$$

for an extended body with mass M orbiting around a central body with mass m. For Mignard tides, where the phase lags are proportional to frequency, the common factor in parentheses is $28\delta_n$. The rate of change of eccentricity is the same in both cases:

$$\frac{1}{e}\frac{de}{dt} = -\frac{3\kappa}{8}\left(3\sin\delta_n + \frac{1}{2}\sin\delta_{2\omega^c - n} - \frac{49}{2}\sin\delta_{2\omega^c - 3n}\right), \quad (243)$$

which again confirms Equation 8 of Goldreich (1963), if his $2\epsilon_3$ is actually the lag for frequency n, not $\frac{3}{2}n$. For Mignard tides, where the phase lags are proportional to frequency,

$$\frac{1}{e}\frac{de}{dt} = -\frac{21}{2}\frac{\kappa}{Q}. \quad (244)$$

15 Tidal friction with eccentricity and obliquity

Next we consider a perturbing body in an elliptic orbit about a dissipative body with obliquity. We continue to assume that there is no wobble ($\omega^a = \omega^b = 0$).

The equations of motion are

$$C'\dot{\omega}^c = \sqrt{3}MR^2\mathcal{F}_1\left(\Pi_{1,1}\beta'\gamma' + \Pi_{1,2}2\alpha'\beta' - \Pi_{1,-1}\alpha'\gamma' + \Pi_{1,-2}((\beta')^2 - (\alpha')^2)\right) \quad (245)$$

with

$$\ddot{\Pi}_{1,0} + d_1\dot{\Pi}_{1,0} + \omega_{21}^2\Pi_{1,0} = -2C_1(\omega^c)^2 + \mathcal{F}_1\frac{2(\gamma')^2 - (\alpha')^2 - (\beta')^2}{2} \quad (246)$$

$$\ddot{\Pi}_{1,1} + d_1\dot{\Pi}_{1,1} + \omega_{21}^2\Pi_{1,1} = \mathcal{F}_1\sqrt{3}\alpha'\gamma' \quad (247)$$

$$\ddot{\Pi}_{1,2} + d_1\dot{\Pi}_{1,2} + \omega_{21}^2\Pi_{1,2} = \mathcal{F}_1\sqrt{3}((\alpha')^2 - (\beta')^2)/2 \quad (248)$$

$$\ddot{\Pi}_{1,-1} + d_1\dot{\Pi}_{1,-1} + \omega_{21}^2\Pi_{1,-1} = \mathcal{F}_1\sqrt{3}\beta'\gamma' \quad (249)$$

$$\ddot{\Pi}_{1,-2} + d_1\dot{\Pi}_{1,-2} + \omega_{21}^2\Pi_{1,-2} = \mathcal{F}_1\sqrt{3}\alpha'\beta' \quad (250)$$

We have again ignored a small periodic time dependence in C'.

The direction cosines are, in this approximation, $\mathbf{o} \cdot \mathbf{s}$ where

$$\mathbf{o} = \mathbf{R}_z(\Omega)\mathbf{R}_x(i)\mathbf{R}_z(\lambda)(1, 0, 0), \tag{251}$$

with $\lambda = f + \omega$, the sum of the true anomaly and the argument of pericenter, and

$$\mathbf{s} = \mathbf{R}_z(\Omega)\mathbf{R}_x(i)\mathbf{R}_z(\Lambda)\mathbf{R}_x(I)\mathbf{R}_z(\phi)\mathbf{s}_0, \tag{252}$$

and $\mathbf{s}_0$ is either $(1, 0, 0)$, $(0, 1, 0)$, or $(0, 0, 1)$, for α', β', and γ', respectively. Here I is the obliquity of the body to the orbit, and Λ is the longitude of the equator minus the longitude of the ascending node of the orbit, and $\phi = \omega^c t$. We find

$$\alpha' = g_0 \cos(\lambda' - \phi) + g_1 \cos(\lambda' + \phi) \tag{253}$$

$$\beta' = g_0 \sin(\lambda' - \phi) - g_1 \sin(\lambda' + \phi) \tag{254}$$

$$\gamma' = g_2 \sin(\lambda'), \tag{255}$$

with $\lambda' = \lambda - \Lambda$, and

$$g_0 = (1 + \cos I)/2 \tag{256}$$

$$g_1 = (1 - \cos I)/2 \tag{257}$$

$$g_2 = -\sin I. \tag{258}$$

In terms of these, we can write down terms that enter the equations of motion

$$\alpha'\beta' = \frac{g_0^2}{2} \sin(2\lambda' - 2\phi) - \frac{g_1^2}{2} \sin(2\lambda' + 2\phi) - g_0 g_1 \sin(2\phi) \tag{259}$$

$$\alpha'\gamma' = \frac{g_0 g_2}{2} \sin(2\lambda' - \phi) + \frac{g_1 g_2}{2} \sin(2\lambda' + \phi)$$
$$+ \frac{g_2(g_0 - g_1)}{2} \sin(\phi) \tag{260}$$

$$\beta'\gamma' = -\frac{g_0 g_2}{2} \cos(2\lambda' - \phi) + \frac{g_1 g_2}{2} \cos(2\lambda' + \phi)$$
$$+ \frac{g_2(g_0 - g_1)}{2} \cos(\phi) \tag{261}$$

$$\frac{(\alpha')^2 - (\beta')^2}{2} = \frac{g_0^2}{2} \cos(2\lambda' - 2\phi) + \frac{g_1^2}{2} \cos(2\lambda' + 2\phi)$$
$$+ g_0 g_1 \cos(2\phi) \tag{262}$$

$$\frac{2(\gamma')^2 - (\alpha')^2 - (\beta')^2}{2} = \frac{g_2^2 - g_0^2 - g_1^2}{2} - \frac{g_2^2 + 2g_0 g_1}{2} \cos(2\lambda'). \tag{263}$$

Proceeding as before, we Fourier expand the right-hand sides using the Hansen functions, and find the solutions of the Π equations under the assumption that the orbital period is much longer than the modal periods:

$$\Pi_{1,1} = \frac{\sqrt{3}}{2} \frac{\mathcal{G}_1}{\omega_{21}^2} \sum_{j=-\infty}^{\infty} \left(- X_j^{-3,2}(e)g_0 g_2 \sin(\phi - jnt - 2(\omega - \Lambda) - \delta_{\omega^c - jn}) \right.$$

$$+ X_j^{-3,2}(e)g_1 g_2 \sin(\phi + jnt + 2(\omega - \Lambda) - \delta_{\omega^c + jn})$$

$$+ \frac{1}{2} X_j^{-3,0}(e)g_2(g_0 - g_1) \sin(\phi + jnt - \delta_{\omega^c + jn})$$

$$\left. + \frac{1}{2} X_j^{-3,0}(e)g_2(g_0 - g_1) \sin(\phi - jnt - \delta_{\omega^c - jn}) \right) \tag{264}$$

$$\Pi_{1,2} = \frac{\sqrt{3}}{2}\frac{\mathcal{G}_1}{\omega_{21}^2}\sum_{j=-\infty}^{\infty}\left(X_j^{-3,2}(e)g_0^2\cos(2\phi - jnt - 2(\omega - \Lambda) - \delta_{2\omega^c - jn})\right.$$

$$+X_j^{-3,2}(e)g_1^2\cos(2\phi + jnt + 2(\omega - \Lambda) - \delta_{2\omega^c + jn})$$

$$+\frac{1}{2}X_j^{-3,0}(e)2g_0g_1\cos(2\phi + jnt - \delta_{2\omega^c + jn})$$

$$\left.+\frac{1}{2}X_j^{-3,0}(e)2g_0g_1\cos(2\phi - jnt - \delta_{2\omega^c - jn})\right) \tag{265}$$

$$\Pi_{1,-1} = \frac{\sqrt{3}}{2}\frac{\mathcal{G}_1}{\omega_{21}^2}\sum_{j=-\infty}^{\infty}\left(-X_j^{-3,2}(e)g_0g_2\cos(\phi - jnt - 2(\omega - \Lambda) - \delta_{\omega^c - jn})\right.$$

$$+X_j^{-3,2}(e)g_1g_2\cos(\phi + jnt + 2(\omega - \Lambda) - \delta_{\omega^c + jn})$$

$$+\frac{1}{2}X_j^{-3,0}(e)g_2(g_0 - g_1)\cos(\phi + jnt - \delta_{\omega^c + jn})$$

$$\left.+\frac{1}{2}X_j^{-3,0}(e)g_2(g_0 - g_1)\cos(\phi - jnt - \delta_{\omega^c - jn})\right) \tag{266}$$

$$\Pi_{1,-2} = \frac{\sqrt{3}}{2}\frac{\mathcal{G}_1}{\omega_{21}^2}\sum_{j=-\infty}^{\infty}\left(-X_j^{-3,2}(e)g_0^2\sin(2\phi - jnt - 2(\omega - \Lambda) - \delta_{2\omega^c - jn})\right.$$

$$-X_j^{-3,2}(e)g_1^2\sin(2\phi + jnt + 2(\omega - \Lambda) - \delta_{2\omega^c + jn})$$

$$-\frac{1}{2}X_j^{-3,0}(e)2g_0g_1\sin(2\phi + jnt - \delta_{2\omega^c + jn})$$

$$\left.-\frac{1}{2}X_j^{-3,0}(e)2g_0g_1\sin(2\phi - jnt - \delta_{2\omega^c - jn})\right) \tag{267}$$

$$\Pi_{1,0} = -2C_1\frac{(\omega^c)^2}{\omega_{21}^2} + \frac{1}{2}\frac{\mathcal{G}_1}{\omega_{21}^2}\sum_{j=-\infty}^{\infty}\left(X_j^{-3,0}(e)(g_2^2 - g_0^2 - g_1^2)\cos(jnt - \delta_{jn})\right.$$

$$\left.-X_j^{-3,2}(e)(g_2^2 + 2g_0g_1)\cos(jnt + 2(\omega - \Lambda) - \delta_{jn})\right) \tag{268}$$

Generalizing Eq. (146), we find the rate of energy dissipation to be

$$\frac{dE}{dt} = \frac{MR^2\mathcal{G}_1^2}{\omega_{21}^2}\sum_{j=-\infty}^{\infty}\left[F_0(X_j^{-3,2}(e))^2 + F_1(X_j^{-3,0}(e))^2\right. \tag{269}$$

$$+ F_2 X_j^{-3,2}(e)X_j^{-3,0}(e)\cos(2(\omega - \Lambda)) \tag{270}$$

$$\left.+ F_3 X_j^{-3,2}(e)X_{-j}^{-3,2}(e)\cos(4(\omega - \Lambda))\right] \tag{271}$$

with

$$F_0 = \frac{3}{4}\left[(g_0g_2)^2(\omega^c - jn)\sin\delta_{\omega^c - jn} + (g_1g_2)^2(\omega^c + jn)\sin\delta_{\omega^c + jn}\right.$$

$$\left.+ (g_0g_0)^2(2\omega^c - jn)\sin\delta_{2\omega^c - jn} + (g_1g_1)^2(2\omega^c + jn)\sin\delta_{2\omega^c + jn}\right]$$

$$+\frac{1}{8}(g_2^2 + 2g_0g_1)^2 jn\delta_{jn} \tag{272}$$

$$F_1 = \frac{3}{8} \big[(g_2(g_0 - g_1))^2 ((\omega^c - jn) \sin \delta_{\omega^c - jn} + (\omega^c + jn) \sin \delta_{\omega^c + jn})$$

$$+ (2g_0 g_1)^2 ((2\omega^c - jn) \sin \delta_{2\omega^c - jn} + (2\omega^c + jn) \sin \delta_{2\omega^c + jn}) \big]$$

$$+ \frac{1}{4} (g_2^2 - g_0^2 - g_1^2)^2 jn \delta_{jn} \tag{273}$$

$$F_2 = \frac{3}{2} \big[g_2^2 (g_1 - g_0)(g_0(\omega^c - jn) \sin \delta_{\omega^c - jn} - g_1(\omega^c + jn) \sin \delta_{\omega^c + jn})$$

$$+ 2g_0^3 g_1 (2\omega^c - jn) \sin \delta_{2\omega^c - jn} + 2g_0 g_1^3 (2\omega^c + jn) \sin \delta_{2\omega^c + jn} \big]$$

$$- \frac{1}{2} (g_2^2 - g_0^2 - g_1^2)(g_2^2 + 2g_0 g_1) jn \sin \delta_{jn} \tag{274}$$

$$F_3 = \frac{3}{64} g_2^4 \big[-4((\omega^c - jn) \sin \delta_{\omega^c - jn} + (\omega^c + jn) \sin \delta_{\omega^c + jn})$$

$$+ ((2\omega^c - jn) \sin \delta_{2\omega^c - jn} + (2\omega^c + jn) \sin \delta_{2\omega^c + jn}) \big]$$

$$+ \frac{9}{32} g_2^4 jn \sin \delta_{jn}. \tag{275}$$

For the special case of synchronous rotation we set $\omega^c = n$. To derive the familiar case, we set $\delta_{k\omega^c - jn} = (k - j)\delta$, and then approximate $\sin \delta$ by $1/Q$. Making the large μ approximation as before, we derive

$$\frac{dE}{dt} = \frac{21}{2} \frac{k_2}{Q} \frac{Gm^2 n R^5}{a^6} \zeta(e, I), \tag{276}$$

where

$$\zeta(e, I) = \frac{2}{7} \bar{f}_0(e) - \frac{4}{7} \bar{f}_1(e) \cos I + \frac{1}{7} \bar{f}_2(e)(1 + (\cos I)^2) \tag{277}$$

$$+ \frac{3}{28} e^2 \bar{f}_3(e)(\sin I)^2 \cos(2(\omega - \Lambda)). \tag{278}$$

with

$$\bar{f}_0(e) = 1 + 23e^2 + 180e^4 + \frac{6765}{8} e^6 + \cdots \tag{279}$$

$$\bar{f}_1(e) = 1 + \frac{27}{2} e^2 + \frac{573}{8} e^4 + \frac{3961}{16} e^6 + \cdots \tag{280}$$

$$\bar{f}_2(e) = 1 + \frac{15}{2} e^2 + \frac{105}{4} e^4 + \frac{525}{8} e^6 + \cdots \tag{281}$$

$$\bar{f}_3(e) = 1 + \frac{14}{3} e^2 + \frac{105}{8} e^4 + \frac{231}{8} e^6 + \cdots . \tag{282}$$

This agrees with Wisdom [2008], where these series are given a finite representation:

$$\bar{f}_0(e) = \frac{f_0(e)}{\beta^{15}}, \tag{283}$$

$$\bar{f}_1(e) = \frac{f_1(e)}{\beta^{12}}, \tag{284}$$

$$\bar{f}_2(e) = \frac{f_2(e)}{\beta^9}, \tag{285}$$

$$\bar{f}_3(e) = \frac{f_3(e)}{\beta^{13}}, \tag{286}$$

where

$$f_3(e) = 1 - \frac{11}{6}e^2 + \frac{2}{3}e^4 + \frac{1}{6}e^6. \tag{287}$$

16 Conclusion

In this paper we have presented a new framework for understanding and calculating the effects of tidal friction on solid bodies. Our approach is to consider the body as a dynamical system described by its modes of oscillation. The dynamics of the modes are determined by the internal distribution of matter and the rheology. Dissipation occurs in the dynamics of the modes. Here we consider only a simple model for dissipation: we assume the modal dynamics is governed by a damped harmonic oscillator-like equation of motion. We derive closed differential equations that govern the tidal process. This is to be contrasted with the usual Darwinian approach in which tides are first raised, then given a phase lag before considering their effects on other bodies. We are able to rederive the Darwin result as an approximate solution to our differential equations. Note that we need not make this approximation—we can directly integrate the differential equations governing the tidal process. We rederive many of the standard results of tidal theory directly in our new framework. We plan to extend these results to different rheologies in a future work.

Appendix: Surface and solid spherical harmonics

We will use the terms "solid spherical harmonic" and "surface spherical harmonic." The normalized surface spherical harmonics are

$$C_l^m(\theta, \phi) = P_l^m(\cos\theta)\cos(m\phi) \tag{288}$$
$$S_l^m(\theta, \phi) = P_l^m(\cos\theta)\sin(m\phi), \tag{289}$$

where the normalized associated Legendre polynomials satisfy

$$P_l^m(x) = N_l^m P_{lm}(x), \tag{290}$$

with

$$N_l^m = \left[(2 - \delta_{m,0})(2l+1)\frac{(l-m)!}{(l+m)!} \right]^{1/2}, \tag{291}$$

and

$$P_{lm}(x) = \frac{1}{2^l l!}(1 - x^2)^{m/2}\frac{d^{l+m}}{dx^{l+m}}\left[(x^2 - 1)^l \right]. \tag{292}$$

A few of the P_{lm} are

$$P_{20}(\cos\theta) = \frac{3}{2}(\cos\theta)^2 - \frac{1}{2} \tag{293}$$
$$P_{21}(\cos\theta) = 3\sin\theta\cos\theta \tag{294}$$
$$P_{22}(\cos\theta) = 3(\sin\theta)^2. \tag{295}$$

For convenience we introduce

$$X_l^m(\theta, \phi) = C_l^m(\theta, \phi) \quad m \geq 0 \tag{296}$$
$$= S_l^{-m}(\theta, \phi) \quad m < 0 \tag{297}$$

The X_l^m (C_l^m and S_l^m) are orthonormal in that

$$\delta_{ll'}\delta_{mm'} = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} X_l^m(\theta,\phi)X_{l'}^{m'}(\theta,\phi)\sin\theta d\theta d\phi \tag{298}$$

The solid spherical harmonics are

$$\tilde{X}_l^m\left(\frac{x}{R},\frac{y}{R},\frac{z}{R}\right) = \left(\frac{r}{R}\right)^l X_l^m(\theta,\phi), \tag{299}$$

where $r^2 = x^2 + y^2 + z^2$, $\cos\theta = z/r$, $\phi = \text{atan}(y,x)$. The solid harmonics satisfy the orthogonality relations

$$\frac{\delta_{ll'}\delta_{mm'}}{2l+3} \tag{300}$$

$$= \frac{1}{4\pi R^3}\int_0^a\int_0^\pi\int_0^{2\pi} \tilde{X}_l^m\left(\frac{x}{R},\frac{y}{R},\frac{z}{R}\right)\tilde{X}_{l'}^{m'}\left(\frac{x}{R},\frac{y}{R},\frac{z}{R}\right)r^2\sin\theta dr d\theta d\phi \tag{301}$$

$$= \frac{1}{4\pi a^3}\int_V \tilde{X}_l^m\tilde{X}_{l'}^{m'}dV, \tag{302}$$

where the V is the sphere of radius R, and the last line introduces an abbreviated notation. Note that whenever we write a $\tilde{X}_l^m$ without arguments, they are assumed to be $(x/R, y/R, z/R)$.

It is convenient to introduce

$$\bar{X}_l^m = \frac{1}{\sqrt{2l+1}}\tilde{X}_l^m. \tag{303}$$

The first few $\bar{X}_l^m$ are

$$\bar{X}_0^0(x,y,z) = 1 \tag{304}$$

$$\bar{X}_1^0(x,y,z) = z \tag{305}$$

$$\bar{X}_1^1(x,y,z) = x \tag{306}$$

$$\bar{X}_1^{-1}(x,y,z) = y \tag{307}$$

$$\bar{X}_2^0(x,y,z) = (2z^2 - x^2 - y^2)/2 \tag{308}$$

$$\bar{X}_2^1(x,y,z) = \sqrt{3}xz \tag{309}$$

$$\bar{X}_2^2(x,y,z) = \sqrt{3}(x^2 - y^2)/2 \tag{310}$$

$$\bar{X}_2^{-1}(x,y,z) = \sqrt{3}yz \tag{311}$$

$$\bar{X}_2^{-2}(x,y,z) = \sqrt{3}xy. \tag{312}$$

Let $\tilde{X}_l^m$ be a solid spherical harmonic. It is a homogeneous function of degree l in the coordinates $(x/R, y/R, z/R)$. Euler's theorem tells us that

$$(\mathbf{x}\cdot\nabla)\tilde{X}_l^m(x/R,y/R,z/R) = l\tilde{X}_l^m(x/R,y/R,z/R). \tag{313}$$

A little calculation shows that

$$\nabla^2(\mathbf{x}\tilde{X}_l^m(x/R,y/R,z/R)) = 2\nabla\tilde{X}_l^m(x/R,y/R,z/R), \tag{314}$$

and

$$\nabla^2(r^k\tilde{X}_l^m(x/R,y/R,z/R)) = k(k+2l+1)r^{k-2}\tilde{X}_l^m(x/R,y/R,z/R). \tag{315}$$

References

Chandler, S.C.: On the variation of latitude. Astron. J. **11**, 59–61 (1891)

Darwin, G.: Scientific Papers, vol. 2. Cambridge University Press, Cambridge (1908)

Dobrovolskis, A.: Spin states and climates of eccentric exoplanets. Icarus **192**, 1–23 (2007)

Eggleton, P., Kiseleva, L., Hut, P.: The equilibrium tide model for tidal friction. Astrophys. J. **499**, 853–870 (1998)

Goldreich, P.: On the eccentricity of satellite orbits in the solar system. MNRAS **126**, 257–268 (1963)

Goldreich, P.: History of the lunar orbit. Rev. Geophys. Space Phys. **4**, 411–439 (1966)

Goldreich, P., Peale, S.J.: Spin-orbit coupling in the solar system. Astron. J. **71**, 425–438 (1966)

Hut, P.: Tidal evolution in close binary systems. Astron. Astrophys. **99**, 126–140 (1981)

Jefferys, H.: The Earth, 6th edn. Cambridge University Press, Cambridge (1976)

Kaula, W.M.: Tidal dissipation by solid friction and the resulting orbital evolution. Rev. Geophys. **2**, 661–685 (1966)

Lamb, H.: On the vibrations of an elastic sphere. Proc. Lond. Math Soc. **13**, 189–212 (1882)

Love, A.E.H.: A Treatise on the Mathematical Theory of Elasticity, 4th edn, pp. 257–259. Dover, New York, NY (1944)

Mignard, F.: The lunar orbit revisited, III. Moon Planets **24**, 189–207 (1981)

Newcomb, S.: On the dynamics of the earth's rotation, with respect to the periodic variations of latitude. MNRAS **52**, 336–341 (1882)

Peale, S.J.: Rotation of solid bodies in the solar system. Rev. Geophys. Space Phys. **11**, 767–793 (1973)

Peale, S.J., Cassen, P.: Contribution of tidal dissipation to lunar thermal history. Icarus **36**, 245–269 (1978)

Peale, S.J., Cassen, P., Reynolds, R.: Melting of Io by tidal dissipation. Science **203**, 892–894 (1979)

Peale, S.J.: Resonances. In: Burns, J.A., Matthews, M.S. (eds.) Satellites, pp. 159–223. University of Arizona Press, Tucson (1986)

Poincaré, H.: Sur une forme novelle des équations de la mécanique. Comptes rendus de l'Académie des Sciences **132**, 369–371 (1901)

Stacey, F.D.: Physics of the Earth, 3rd edn. Brookfield Press, Brisbane (1992)

Williams, J.G.: Lunar core and mantle. What does LLR see? Proceedings of the 16th International Workshop on Laser Ranging, pp. 101–120 (2008)

Wisdom, J.: Tidal heating at arbitrary eccentricity and obliquity. Icarus **193**, 637–640 (2008)

Zahn, J.-P.: The dynamical tide in close binaries. Astron. Astrophys. **41**, 329–344 (1975)