

RESOLVING THE PERICENTER

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ABSTRACT

The Wisdom–Holman mapping method and its variations have become a mainstay of research in solar system dynamics. But the method is not without its limitations. Rauch & Holman noted that at large eccentricities sufficiently small steps must be taken to resolve the pericenter. In this paper, I explore in more detail what it means to resolve the pericenter.

Key words: celestial mechanics – methods: numerical

1. INTRODUCTION

The Wisdom–Holman mapping method (Wisdom & Holman 1991) and its variations have become a mainstay of research in solar system dynamics. We used Jacobi coordinates to effect the elimination of the center of mass and the splitting of the Hamiltonian for the n -body problem into Kepler Hamiltonians plus perturbations. Touma & Wisdom (1993a, 1993b) used the canonical heliocentric variables to split the Hamiltonian into Kepler plus perturbations. Duncan et al. (1998) and Chambers (1999) used the Wisdom–Holman method in “democratic heliocentric” variables, the canonical heliocentric variables with a slightly different splitting. Close encounters among planets are treated specially in these works. Chambers (1999) switched to the conventional Bulirsch–Stoer method during an encounter; Duncan et al. (1998) recursively subdivide the stepsize. Any Wisdom–Holman method written in democratic heliocentric variables performs poorly at large eccentricity unless special care is used to integrate “close encounters with the Sun” (Levison & Duncan 2000). Thus the “Mercury” code (Chambers 1999) should not be used if any of the eccentricities of the planets becomes large. The original “Symba” code (Duncan et al. 1998) also had this problem and should not be used at large eccentricity, but it has been modified to fix the problem by switching to the Bulirsch–Stoer algorithm if a planet is close to the Sun (Levison & Duncan 2000). The original Wisdom–Holman method in Jacobi coordinates (Wisdom & Holman 1991) does not have this problem at large eccentricity. But it does exhibit “stepsize resonances” (Wisdom & Holman 1992) that limit the stepsize to roughly $1/20$ of the orbital period of the innermost planet, seen clearly when using the symplectic corrector (Wisdom et al. 1996). Rauch & Holman (1999) found that at large eccentricity stepsize resonances further limit the stepsize used in the Wisdom–Holman method so that the pericenter is resolved. This paper explores further what is required to resolve the pericenter in two standard test problems: the Stark problem used by Rauch & Holman (1999) and the two-planet problem used by Levison & Duncan (2000).

2. STARK PROBLEM

The Hamiltonian for the three-dimensional Stark problem, as defined by Rauch & Holman (1999), is

$$H(t, x, p) = \frac{p^2}{2} - \frac{GM}{r} - S \cdot x, \quad (1)$$

where S is the Stark vector. I make a mapping for this problem with the splitting

$$H_A(t, x, p) = \frac{p^2}{2} - \frac{GM}{r}, \quad (2)$$

the Kepler problem with “Kepler constant” GM , and

$$H_B(t, x, p) = -S \cdot x. \quad (3)$$

The three-dimensional Stark problem exhibits oscillations in eccentricity with a peak eccentricity. Thus it is a nice context to explore stepsize resonances that become important at large eccentricity. I use $S_x = 2.75 \times 10^{-5}$, $S_y = 0$, and $S_z = 2.75 \times 10^{-5}$, with $G = (0.0172)^2$ (units solar mass, day, AU), and $M = 1$. The initial semimajor axis is 0.4 AU, and the initial eccentricity is 0.8. I adjust the peak eccentricity by varying the initial inclination. The integrations span 100,000 days, or roughly 1000 orbits.

In addition to studying stepsize resonances, the Stark problem is also a stringent test for the “Kepler solver.” The Kepler solver advances the equivalent Kepler problem using Gauss’s f and g functions, as recommended by Wisdom & Holman (1991). Here I use a Kepler solver written in universal variables (Wisdom & Hernandez 2015). By being careful with the formulation we are able to avoid the calculation of the Stumpff series (and argument four-folding).

Figures 1–3 show results for the Stark problem using our universal variable Kepler solver. The symplectic corrector (Wisdom et al. 1996) is able to correct the results whenever the error is dominated by truncation error. Apparently, for the Stark problem, a good rule of thumb is that we need at least approximately 17 steps per T_f , where

$$T_f = \frac{2\pi}{\dot{f}_{\max}}, \quad (4)$$

is the effective period at pericenter, and where \dot{f}_{\max} is the maximum rate of change of the true anomaly (at pericenter). This is what it means to “resolve the pericenter.”

3. TWO-PLANET PROBLEM

Duncan et al. (1998) and Levison & Duncan (2000) explore a two-planet problem to compare the accuracy of their methods. They also compare their results to results obtained with their implementation of the Wisdom–Holman method, which they call the MVS method. They explore a modified Jupiter–Saturn problem, with the inclination of Saturn set at

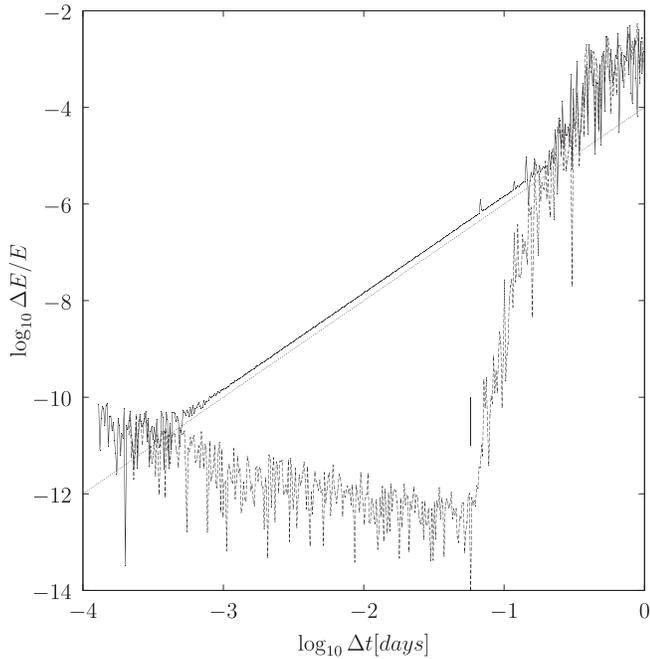


Figure 1. Initial inclination of 50° leads to a maximum eccentricity of approximately 0.94. Three traces are shown. The uppermost curve (solid line) shows the results without the symplectic corrector. There are three regimes: (1) the noisy part of the curve on the upper right is a regime dominated by stepsize resonances, (2) the middle part of the curve is dominated by truncation error, and (3) the noisy part on the lower left is dominated by roundoff error. The diagonal dotted line has a slope of 2. The lower curve (dashed line) shows the results with the symplectic corrector. The symplectic corrector removes the interval dominated by truncation error. The border between the stepsize resonance region and the roundoff error region in the corrected curve has approximately 17 steps per T_j , as indicated by the vertical stroke.

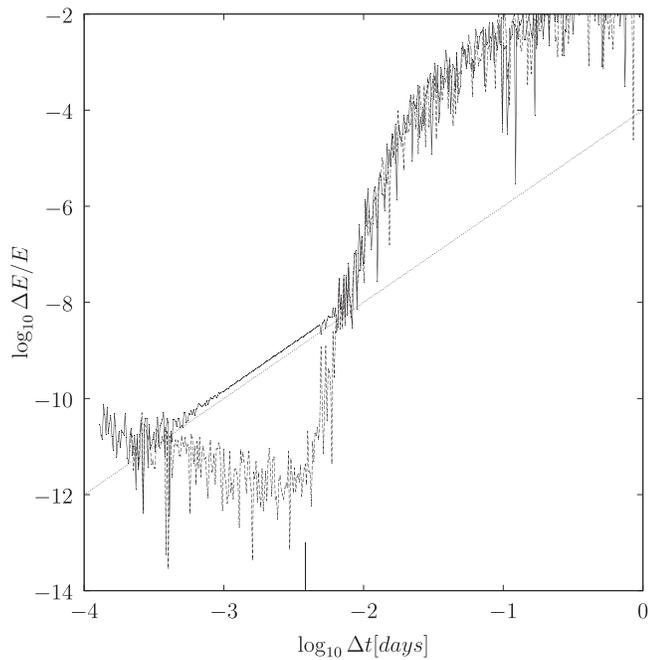


Figure 2. Initial inclination of 75° leads to a maximum eccentricity of approximately 0.99. The curves are as described in Figure 1. The border between the stepsize dominated regime and the roundoff error regime in the corrected curve is again approximately 17 steps per T_j , as indicated by the vertical stroke.

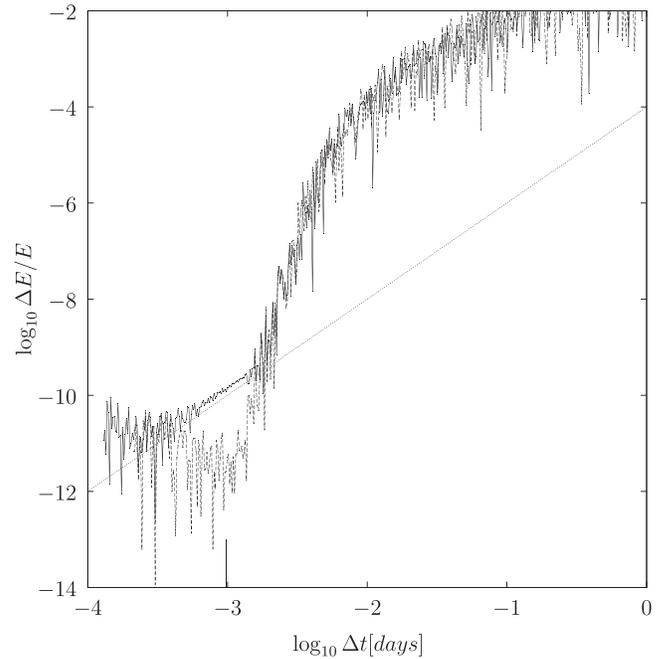


Figure 3. Initial inclination of 100° leads to a maximum eccentricity of approximately 0.996. The curves are as described in Figure 1. The border between the stepsize dominated regime and the roundoff error regime in the corrected curve is again approximately 17 steps per T_j , as indicated by the vertical stroke.

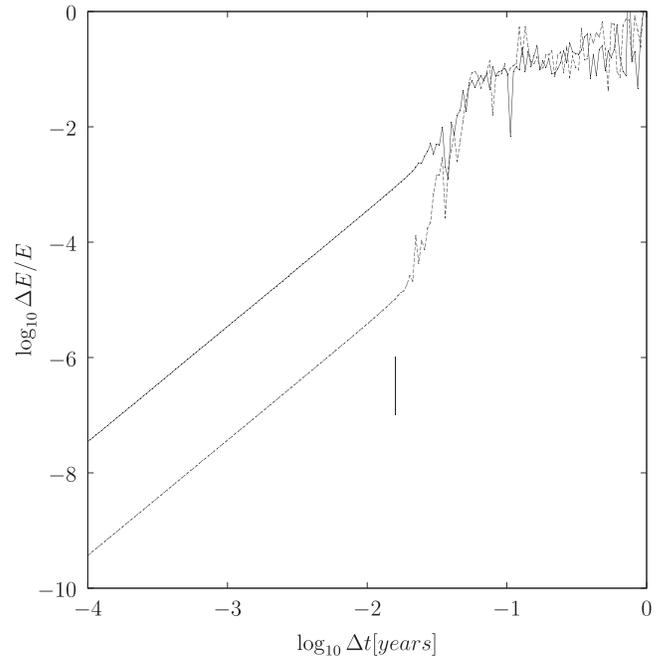


Figure 4. Logarithm of the relative energy error is plotted versus the logarithm of the timestep (in years) for the two-planet problem. The region in the upper right is dominated by stepsize resonances. The rest is dominated by truncation error. The upper curve (solid line) does not use the corrector; the lower curve (dashed line) uses the corrector. The vertical stroke indicates the stepsize for 15 steps per effective pericenter period.

$\pi/2$, and vary the initial eccentricity of Saturn. They take a stepsize of 0.15 year. They report that the DH method (Duncan et al. 1998) and the Wisdom–Holman method are both “unusable” in this problem for perihelion distances less than

~ 1 AU. The failure of the DH method is easy to understand, at high eccentricities the mapping in democratic heliocentric coordinates does not integrate Keplerian orbits well, even without interactions (Levison & Duncan 2000). Regarding the Wisdom–Holman method, Duncan et al. (1998, p. 2072–2073) state “the [Wisdom–Holman method] can only handle small pericentric distances for massless particles and for the *innermost* planet.” They go on: “For a massive body that is not the innermost planet, ... the [Wisdom–Holman method] will fail.” Their reasoning is mysterious and their conclusion is incorrect. Though they mention the paper of Rauch and Holman, they do not attach significance to it. In fact, the Wisdom–Holman method fails in their test case because they chose a stepsize that did not resolve the pericenter. I demonstrate this by redoing their experiment with an appropriate range of stepsizes.

I take $G = (0.0172)^2$, for which the units are days, AU, and solar mass. I take the solar mass to be unity. The mass of “Jupiter” is $1/1047.355$ and the mass of “Saturn” is $1/3498.5$. The semimajor axis of Jupiter is 5.2AU; that of Saturn is 9.58 AU. The eccentricity of Jupiter is 0.05. The eccentricity of Saturn is varied. The inclination of Jupiter is zero; that of Saturn is $\pi/2$. The argument of pericenter of Jupiter and Saturn are zero, as are the longitudes of ascending node. The integrations span 3000 years.

Figure 4 shows the relative error in the two-planet problem as a function of stepsize. The initial eccentricity of Saturn is 0.95. We see two regions: a region in the upper right that is dominated by stepsize resonances, and the rest is dominated by

truncation error. The vertical stroke indicates a stepsize that is $1/15$ of T_f , the effective pericenter period. We see that as long as the stepsize is chosen to resolve the pericenter, the Wisdom–Holman method has no trouble evolving this two-planet problem, contrary to the conclusions of Levison & Duncan (2000).

4. CONCLUSION

Numerical exploration of the Stark problem and the two-planet problem indicates the pericenter is adequately resolved if the stepsize is chosen to be $1/20$ of the effective period at pericenter. Depending on the problem, the error may or may not be dominated by truncation error at this stepsize.

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