IMO Training Camp Buffet Contest

June 30, 2008

1. Let A be a subset of $\{1, 2, \ldots, 2008\}$, such that for all $x, y \in A$ with $x \neq y$, the sum $x + y$ is not divisible by 1004. Find, with proof, the maximum possible size of A.

   Solution: We can group the 2008 numbers in 1003 pairs of the form $\{k, 2008-k\}$, with $k \in \{1, 2, \ldots, 1003\}$, and the pair $\{1004, 2008\}$. Observe that A cannot contain two elements from the same pair, as they would add up to a number divisible by 1004. Thus $|A| \leq 1004$.

   To see that $|A| = 1004$ is possible, construct A by taking all the elements of $\{1, 2, \ldots, 2008\}$ whose remainder upon division by 1004 is less than or equal to 502, and removing the elements 1506 and 2008 from the resulting set. This yields a 1004-element set A satisfying the desired properties (why?).

2. Find, with proof, all real number solutions to the following:

   $$(a^2+1)(b^2+1) = (ab+1)(a+b).$$

   Solution: We have

   $$\begin{aligned} & 2(a^2+1)(b^2+1) - 2(ab+1)(a+b) \\ &= 2(a^2b^2 + a^2 + b^2 + 1 - a^2b - ab^2 - a - b) \\ &= (a^2b^2 - 2a^2b + a^2) + (a^2b^2 - 2ab^2 + b^2) + (a^2 - 2a + 1) + (b^2 - 2b + 1) \\ &= a^2(b-1)^2 + b^2(a-1)^2 + (a-1)^2 + (b-1)^2 \\ &= (b^2+1)(a-1)^2 + (a^2+1)(b-1)^2, \end{aligned}$$

   which is positive unless $a = 1$ and $b = 1$, which forms a solution. Therefore, the only solution is $a = b = 1$.

   Remark: Alternatively, note that by Cauchy-Schwarz inequality, we have

   $$(a^2+1)(b^2+1) \geq (ab+1)^2 \qquad \text{and} \qquad (a^2+1)(1+b^2) \geq (a+b)^2$$

   and multiplying together yields $[(a^2+1)(b^2+1)]^2 \geq [(ab+1)(a+b)]^2$, so we can find all the solutions to the equation by consider the equality cases in the above inequalities.

3. Find all ordered pairs (x, y) of positive integers such that $2^x = 3^y + 7$.

   Solution: In mod 3, we have $(-1)^x \equiv 1 \pmod 3$, so that x is even. Let $x = 2a$. Now, in mod 4, we have $0 \equiv (-1)^y - 1 \pmod 4$, so that y is even. Let $y = 2b$. Then $2^{2a} - 3^{2b} = 7$, so that

   $$(2^a + 3^b)(2^a - 3^b) = 7.$$

   Since $2^a + 3^b > 0$, $2^a + 3^b > 2^a - 3^b$, and 7 is prime, we must have $2^a + 3^b = 7$ and $2^a - 3^b = 1$. So $2^a = 4$ and $3^b = 3$. Thus $(a, b) = (2, 1)$ and so $(x, y) = (2a, 2b) = (4, 2)$.

4. To *clip* a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon P_8, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is at least $\frac{1}{2}$, for all $n \geq 6$.

   Solution: The key observation is that for any $n \geq 6$ and any side of $\mathcal{P}_6$, some subsegment of this side is a side of $\mathcal{P}_n$ (this can be easily proven using induction). So, for any $\mathcal{P}_n$, we can select points $P_1, P_2, \ldots, P_6$ on its perimeter so that P_i lies on the i-th side of $\mathcal{P}_6$. Since $\mathcal{P}_n$ is convex, it contains the hexagon $P_1P_2P_3P_4P_5P_6$. Therefore, it suffices to prove that the area of $P_1P_2\dots P_6$ is at least $\frac{1}{2}$ whenever P_i lies on the i-th side of $\mathcal{P}_6$ for each i.

Consider this problem as a minimization problem, where we want to minimize the area of $P_1P_2P_3P_4P_5P_6$ subject to the above condition. Observe that as P_i moves along the i-th side of $\mathcal{P}_6$, the area of $P_1P_2P_3P_4P_5P_6$ changes monotonically (in fact, linearly) as P_i moves form one end to the other. Therefore, the minimum must occur when P_i coincides with a vertex of $\mathcal{P}_6$. Therefore, we simply needs to search through the set of (possibly degenerate) hexagons $P_1P_2P_3P_4P_5P_6$ with the property that each P_i is one of the endpoints of the i-th side of $\mathcal{P}_6$. We wish to find the one with the minimum area. After some work, we see that the minimum occurs when $P_1P_2P_3P_4P_5P_6$ is an equilateral triangle, and its area is $\frac{1}{2}$.

Therefore, the area of $\mathcal{P}_n$ is at least $\frac{1}{2}$.

Remark: More elegantly, the bound can be proven using the inequality

$$\begin{aligned}
&x_1(1-x_2)+x_2(1-x_3)+x_3(1-x_4)+x_4(1-x_5)+x_5(1-x_6)+x_6(1-x_1)\\
&\le (1-x_2)+x_2+(1-x_4)+x_4+(1-x_6)+x_6\\
&\le 3.
\end{aligned}$$

I'll leave the details to you.

Also, the bound $\frac{1}{2}$ is not optimal. Can you find a better bound? What's the best bound? (I don't know the answer to the last question.)

5. Let n be a positive integer. Suppose that $\theta_1, \theta_2, \dots, \theta_n$ are angles with $0 < \theta_i < \frac{\pi}{2}$ for each i such that

$$\cos^2\theta_1 + \cos^2\theta_2 + \cdots + \cos^2\theta_n = 1.$$

Prove that

$$\tan\theta_1 + \tan\theta_2 + \cdots + \tan\theta_n \ge (n-1)(\cot\theta_1 + \cot\theta_2 + \cdots + \cot\theta_n).$$

Solution: Let $a_i = \cos\theta_i$. Then

$$\begin{aligned}
\tan\theta_i = \frac{\sin\theta_i}{\cos\theta_i} = \frac{\sqrt{1-\cos^2\theta_i}}{\cos\theta_i} &= \frac{\sqrt{a_1^2+a_2^2+\cdots+a_{i-1}^2+a_{i+1}^2+\cdots+a_n^2}}{a_i}\\
&\ge \frac{a_1+a_2+\cdots+a_{i-1}+a_{i+1}+\cdots+a_n}{a_i\sqrt{n-1}}
\end{aligned}$$

by the Power-Mean inequality. Summing the above inequality for $i = 1, 2, \dots, n$ yields

$$\sum_{i=1}^{n} \tan\theta_i \ge \frac{1}{\sqrt{n-1}} \sum_{i=1}^{n}\sum_{j\ne i} \frac{a_j}{a_i} = \sqrt{n-1}\sum_{i\ne j}\frac{a_j}{a_i},$$

as each ratio $\frac{a_i}{a_j}$ appears $n-1$ times.

On the other hand, we have

$$\begin{aligned}
\cot\theta_i = \frac{\cos\theta_i}{\sin\theta_i} = \frac{\cos\theta_i}{\sqrt{1-\cos^2\theta_i}} &= \frac{a_i}{\sqrt{a_1^2+a_2^2+\cdots+a_{i-1}^2+a_{i+1}^2+\cdots+a_n^2}}\\
&\le \frac{1}{(n-1)^{3/2}} a_i \left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots\frac{1}{a_{i-1}}+\frac{1}{a_{i+1}}+\cdots\frac{1}{a_n}\right)
\end{aligned}$$

again by the Power-Mean inequality. Summing the above inequalities for $i = 1, 2, \dots, n$ yields

$$\sum_{i=1}^{n} \cot\theta_i \le \frac{1}{(n-1)^{3/2}} \sum_{i=1}^{n}\sum_{j\ne i} \frac{a_j}{a_i} = \sqrt{n-1}\sum_{i\ne j}\frac{a_j}{a_i}.$$

Therefore,

$$\sum_{i=1}^{n} \tan\theta_i \ge \sqrt{n-1}\sum_{i\ne j}\frac{a_j}{a_i} \ge (n-1)\sum_{i=1}^{n}\cot\theta_i.$$

6. Let AA_1, BB_1, CC_1 be the altitudes of an acute triangle ABC. Let O be an arbitrary point inside $A_1B_1C_1$. Denote the feet of the perpendiculars from O to the lines AA_1 and BC by M and N, respectively; the ones from O to the lines BB_1 and CA by P and Q, respectively; the ones from O to the lines CC_1 and AB by R and S, respectively. Prove that the lines MN, PQ, and RS are concurrent.

   Solution: Consider a dilation about O with ratio 2. Let M', P', R' be the image of M, P, R respectively. The dilation sends line MN to line A_1M', line PQ to line B_1P', line RS to line C_1R'. So it suffices to prove that A_1M', B_1P', C_1R' are concurrent.

   Observe that in triangle $A_1B_1C_1$, A_1A is an angle bisector (why?), and $\angle M'A_1A = \angle MNO = \angle OA_1A$. That is, line A_1M' is the reflection of A_1O across the angle bisector of $\angle B_1A_1C_1$. Therefore, A_1M', B_1P', C_1R' concur at the isogonal conjugate of O with respect to triangle $A_1B_1C_1$.

7. Positive integers a and b are given such that $2a+1$ and $2b+1$ are relatively prime. Find all possible values of the greatest common divisor of $2^{2a+1}+2^{a+1}+1$ and $2^{2b+1}+2^{b+1}+1$.

   Solution: We begin with a well-known lemma:

   Lemma. For any positive integers k and n, we have $\gcd(2^k-1, 2^n-1) = 2^{\gcd(k,n)}-1$.

   Proof. We use induction on the quantity $k+n$. If $k+n=1$, the claim is obvious. Now, using the fact that $\gcd(a,b) = \gcd(a-b,b)$, we have (for $k \geq n$)

$$\begin{aligned}\gcd(2^k-1, 2^n-1) &= \gcd(2^k-2^n, 2^n-1) = \gcd(2^n(2^{k-n}-1), 2^n-1)\\ &= \gcd(2^{k-n}-1, 2^n-1) = 2^{\gcd(k-n,n)}-1 = 2^{\gcd(k,n)}-1.\end{aligned}$$

   where we applied the induction hypothesis to the pair $(k-n, n)$. □

   Recall the well-known factorization identity

$$4x^4+1 = (2x^2+1)^2 - 4x^2 = (2x^2+2x+1)(2x^2-2x+1).$$

   Applying it to $x = 2^a$ gives us

$$2^{4a+2}+1 = (2^{2a+1}+2^{a+1}+1)(2^{2a+1}-2^{a+1}+1).$$

   It follows that the greatest common divisor d of $2^{2a+1}+2^{a+1}+1$ and $2^{2b+1}+2^{b+1}+1$ divides the greatest common divisor of $2^{4a+2}+1$ and $2^{4b+2}+1$, and hence it divides the greatest divisor of $2^{8a+4}-1$ and $2^{8b+4}-1$, which, by the lemma, equals to $2^{(8a+4,8b+4)}-1 = 2^4-1 = 15$, as we are given that $2a+1$ and $2b+1$ are relatively prime. Therefore, d divides 15.

   Since $2^{2a+1}+2^{a+1}+1 \equiv 2^{a+1} \not\equiv 0 \pmod 3$, 3 does not divide d. Hence $d=1$ or $d=5$. Both cases are possible. For $a=4, b=8$ we obtain $d=5$, and for $a=2$, $b=3$ we obtain $d=1$.

   Second solution: Suppose a prime p divides $2^{2a+1}+2^{a+1}+1$ and $2^{2b+1}+2^{b+1}+1$. Then $2^{2a+1} \equiv 2^{a+1}+1 \pmod p$. Squaring gives

$$2^{4a+2} \equiv 2^{2a+2}+2^{a+2}+1 \equiv 2(2^{2a+1}+2^{a+1}+1)-1 \equiv -1 \pmod p.$$

   Similarly, $2^{4b+2} \equiv -1 \pmod p$. Since $2a+1$ and $2b+1$ are relatively prime, there are integers j, k such that $j(2a+1)+k(2b+1)=1$ (where j and k necessarily have opposite parity). Hence, we get

$$2^2 \equiv 2^{2((2a+1)j+(2b+1)k)} \equiv (-1)^{j+k} \equiv -1 \pmod p.$$

   Hence, $p=5$.

   On the other hand, suppose 25 divides $2(2^{2x+1}+2^{x+1}+1) = (2^{x+1}+1)^2+1$. Since the only solutions to $y^2+1 \equiv 0 \pmod{25}$ are $y \equiv \pm 7 \pmod{25}$, we must have $2^{x+1}+1 \equiv \pm 7 \pmod{25}$. But it is easily checked that 2 is a primitive root mod 25, as its order is not $\phi(25)/2 = 10$ or $\phi(25)/5 = 4$. It follows that there is there a unique solution in mod 20 for x for each of $2^{x+1}+1 \equiv 7 \pmod{25}$ and $2^{x+1}+1 \equiv -7$

(mod 25), namely $x \equiv 7 \pmod{20}$ for the former and $x \equiv 12 \pmod{20}$ for the latter. In either case, we have $5 \mid 2x+1$. This means that if 25 divides the greatest common divisor of $2^{2a+1}+2^{a+1}+1$ and $2^{2b+1}+2^{b+1}+1$, then 5 must divide both $2a+1$ and $2b+1$, contradicting the hypothesis that they are relatively prime.

It follows that the only possible values for the greatest common divisor are 1 and 5. They can be achieved by $(a,b)=(4,8)$ and $(2,3)$ respectively.

Source: St. Petersburg 2002

8. Let X be a finite set, and suppose $A_1,\ldots,A_m$ and $B_1,\ldots,B_m$ are subsets of X with $|A_i|=r$ and $|B_i|=s$ for each i, such that $A_i \cap B_i = \emptyset$ for every i and $A_i \cap B_j \neq \emptyset$ whenever $i \neq j$. Prove that $m \leq \binom{r+s}{r}$.

Solution: (by Béla Bollobàs in 1965) Let $|X|=n$. There are $n!$ ways to label the elements of X with $\{1,2,\ldots,n\}$. Let us pick a random labeling (i.e. permutation), so that every labeling is equally likely to be chosen. For $1 \leq i \leq m$, let E_i denote the event that the highest label in A_i is less than the lowest label in B_i (which we denote by $A_i < B_i$).

First, observe that $\Pr(E_i) = \binom{r+s}{r}^{-1}$ (why?). Now, we claim that if $i \neq j$, then $E_i \cap E_j = \emptyset$. That is, no labeling of X will result in both $A_i < B_i$ and $A_j < B_j$. Indeed, if this were the case, and, say, the highest label of A_i is at least as great as the highest label of A_j (we may switch i and j needed), then we must have $A_j < B_i$ so that $A_i \cap B_j = \emptyset$, contradiction.

Thus, $E_1, E_2, \ldots, E_m$ are mutually disjoint events. It follows that

$$1 \geq \Pr(E_1 \cup E_2 \cup \cdots \cup E_m) = \Pr(E_1) + \Pr(E_2) + \cdots + \Pr(E_m) = m\binom{r+s}{r}^{-1}.$$

Therefore, $|X| = m \leq \binom{r+s}{r}$.