IMO Training Camp Buffet Contest

June 30, 2008

Format of contest:

- At the beginning, every student will be given three problems. IMO team members will be given problems #3, 4, 5, and local students will be given problems #1, 2, 3.
- When a student solves a problem, he/she should write it up and submit it to the graders. When the solution is considered correct (i.e., worthy of a 7), the student will be given a new problem. If the student has already received the last problem in the list, then no new problem will be given.
- If the student's solution contains a minor error, the error will be indicated and the student will be asked to correct it and resubmit the solution.
- The time limit is 4.5 hours.

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1. Let A be a subset of $\{1, 2, \ldots, 2008\}$, such that for all $x, y \in A$ with $x \neq y$, the sum $x + y$ is not divisible by 1004. Find, with proof, the maximum possible size of A.

2. Find, with proof, all real number solutions to the following:

$$(a^2 + 1)(b^2 + 1) = (ab + 1)(a + b).$$

3. Find all ordered pairs (x, y) of positive integers such that $2^x = 3^y + 7$.

4. To *clip* a convex n-gon means to choose a pair of consecutive sides AB, BC and to replace them by the three segments AM, MN, and NC, where M is the midpoint of AB and N is the midpoint of BC. In other words, one cuts off the triangle MBN to obtain a convex $(n+1)$-gon. A regular hexagon $\mathcal{P}_6$ of area 1 is clipped to obtain a heptagon $\mathcal{P}_7$. Then $\mathcal{P}_7$ is clipped (in one of the seven possible ways) to obtain an octagon P_8, and so on. Prove that no matter how the clippings are done, the area of $\mathcal{P}_n$ is at least $\frac{1}{2}$, for all $n \geq 6$.

5. Let n be a positive integer. Suppose that $\theta_1, \theta_2, \ldots, \theta_n$ are angles with $0 < \theta_i < \frac{\pi}{2}$ for each i such that

$$\cos^2\theta_1 + \cos^2\theta_2 + \cdots + \cos^2\theta_n = 1.$$

Prove that

$$\tan\theta_1 + \tan\theta_2 + \cdots + \tan\theta_n \geq (n-1)(\cot\theta_1 + \cot\theta_2 + \cdots + \cot\theta_n).$$

6. Let AA_1, BB_1, CC_1 be the altitudes of an acute triangle ABC. Let O be an arbitrary point inside $A_1B_1C_1$. Denote the feet of the perpendiculars from O to the lines AA_1 and BC by M and N, respectively; the ones from O to the lines BB_1 and CA by P and Q, respectively; the ones from O to the lines CC_1 and AB by R and S, respectively. Prove that the lines MN, PQ, and RS are concurrent.

7. Positive integers a and b are given such that $2a+1$ and $2b+1$ are relatively prime. Find all possible values of the greatest common divisor of $2^{2a+1} + 2^{a+1} + 1$ and $2^{2b+1} + 2^{b+1} + 1$.

8. Let X be a finite set, and suppose $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ are subsets of X with $|A_i| = r$ and $|B_i| = s$ for each i, such that $A_i \cap B_i = \emptyset$ for every i and $A_i \cap B_j \neq \emptyset$ whenever $i \neq j$. Prove that $m \leq \binom{r+s}{r}$.