Algebraic Number Theory

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1. **Challenging Warmup:** A natural number N is called *automorphic* if N^2 ends with N when written out in its decimal notation. For example, 5 is automorphic. Find all automorphic numbers.

2. Find all triples (a, m, n) of natural numbers satisfying $(a^m + 1)|(a + 1)^n$

3. $f : Q \to Q$ is a function which satisfies $f(q + \frac{13}{42}) + f(q) = f(q + \frac{1}{7}) + f(q + \frac{1}{6})$ for all rational numbers q, and $|f(q)| \leq 1$. Prove that f is periodic, i.e. $f(q+c) = f(q)$ for some non-zero c.

4. Prove that for $n > 1$, $\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + ... \frac{x^2}{2} + x + 1 = 0$ does not have rational solutions.

5. Define a_n Recursively by $\sum_{d|n} a_d = 2^n$. Prove that $n|a_n$.

6. Find all natural numbers x, y such that $x^{y^2} = y^x$

7. b, m, n are natural numbers such that $b^n - 1$ and $b^m - 1$ have the same prime factors. Prove that $b - 1$ is a power of 2.

8. Let m, n be relatively primes natural numbers. Call a number representable if it can be written as $am + bn$ for non-negative a, b. Prove that $mn - m - n$ is not representable, but if x is any non-representable number, then $mn - m - n = am + nb + cx$ for non-negative a, b, c.

9. Let $p_1, p_2, ..., p_n$ be distinct prime numbers ≥ 5. Prove that $2^{p_1 p_2 p_3 \cdots p_n} + 1$ has at least 4^n factors.