The theory of Numbers

June 30, 2008

Jacob Tsimerman

Fundamental Theorem of Arithmetic: Every $n \geq 1$ Can be expressed in a unique way as

$$\prod_{1 \leq (i) \leq (k)} p_i^{a_i} = n \text{ With } p_i \text{ prime and } a_i \geq 0.$$

Fermat's little Theorem: For p prime and $(a, p) = 1, a^{p-1} = 1 \pmod p$
Wilson's Theorem: $(p-1)! = -1 \pmod p$

PROBLEMS TO TRY

1. Prove that the sum of the squares of 3 or 4 consecutive integers cannot be a perfect square.
2. Prove that the sequence 1,11,111,... contains an infintie subsequence of relatively prime numbers.
3. Find all real x such that $x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor = 88$
4. You have 2000 identical-looking balls, 1000 of which weigh 10 grams and 1000 of which way 9.9 grams. You are also given a balance, which displays the weight of the Right side minus the weight of the left side. How many weighings do you need to be able to display two sets of balls, each having the same number of balls, that don't weigh the same?
5. If m and n are positive integers, prove $\sum_{i=0}^{n-1} \frac{1}{m+i}$ is not an integer
6. find the largest n such that for every $y < n^{\frac{1}{2}}$, $y|n$.
7. Find the largest n such that for every $y < n^{\frac{1}{3}}$, $y|n$
8. Prove that there exist infinitely many n such that n^4+1 has a prime divisor greater than $2n$.
9. Find the smallest n with $2^{1995}|3^n - 1$
10. $(m, p(p-1)) = 1$, where p is prime and m is a positive integer. Given that a, b are positive integers relatively prime to p, prove:
    (a) $a^m = b^m \mod p$ if and only if $a = b \mod p$
    (b) $a^m = b^m \mod p^2$ if and only if $a = b \mod p^2$
    (c) $a^m = b^m \mod p^{10000}$ if and only if $a = b \mod p^{10000}$

11. If 5^n and 2^n both start with the same digit, what must that digit be?

12. Find all pairs of positive integers k, n such that $7^k - 3^n$ divides $k^4 + n^2$.

13. For a positive integer n, $r(n)$ is defined to be the sum of the remainders upon dividing n by $1, 2, 3, ..., n$. Prove that there are infinitely many n with $r(n) = r(n+1)$.

14. For a positive integer n, $s(n)$ is defined to be $\sum_{i=1}^{n} \lfloor \frac{n}{i} \rfloor$. Find all n with $s(n) = s(n+1)$.

15. Vivek and Margret are playing a game as follows. First, Vivek picks a positive integer greater than 1. On the next move, Margret picks a number not divisible by Vivek's number. On the next move, Vivek has to pick a number that cannot be expressed as a sum of numbers which have already been picked (not neccessarily distinct sum!), and so on. The loser is the person who picks the number 1. Prove that the game will eventually end, and determine who has a winning strategy.

16. Let $a = \sqrt{2}$. Is the last digit of $\lfloor a^n \rfloor$ Periodic?

17. A positive integer a is called *Automorphic* if a^2 ends with the a in its decimal expansion. For example. 5 and 25 are automorphic. Find ALL automorphic numbers.

18. Is there an n divisible by EXACTLY 2008 primes such that $\frac{2^n+1}{n}$ is an integer?

19. $a_0, a_1, ..., a_n$ are positive integers with $a_0 < a_1 < ... < a_n$. Prove
$\frac{1}{lcm(a_0,a_1)} + \frac{1}{lcm(a_1,a_2)} + ... + \frac{1}{lcm(a_{n-1},a_n)} \leq 1 - \frac{1}{2^n}$

20. Find all functions $f : Z \to Z$ such that for all integers m, n, $f(m^2+n) = f(n^2+m)$

21. Find all ordered triples of primes (p, q, r) such that
$p|q^r + 1, q|r^p + 1, r|p^q + 1$.