IMO Training Camp Mock Olympiad #2 Solutions

July 3, 2008

Time limit: 4.5 hours

1. Given an isosceles triangle ABC with $AB = AC$. The midpoint of side BC is denoted by M. Let X be a variable point on the shorter arc MA of the circumcircle of triangle ABM. Let T be the point in the angle domain BMA, for which $\angle TMX = 90^\circ$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on X.

   Solution: See IMO Shortlist 2007 Problem G2

2. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

   $$f(x + f(y)) = f(x + y) + f(y)$$

   for all $x, y \in \mathbb{R}^+$. (Symbol $\mathbb{R}^+$ denotes the set of all positive real numbers.)

   Solution: See IMO Shortlist 2007 Problem A4

3. For a prime p and a positive integer n, denote by $\nu_p(n)$ the exponent of p in the prime factorization of n. Given a positive integer d and a finite set $\{p_1, \ldots, p_k\}$ of primes. Show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n!)$ for all $1 \le i \le k$.

   Solution: See IMO Shortlist 2007 Problem N7

IMO Training Camp Mock Olympiad #3

July 5, 2008

Time limit: 4.5 hours

1. A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that in this dissection, there exists a rectangle having no point on the boundary of the square.

   Solution: See IMO Shortlist 2007 Problem C2

2. The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

   Solution: See IMO Shortlist 2007 Problem G3

3. Let n be a fixed positive integer. Find the maximum value of the expression

$$\frac{(ab)^n}{1-ab} + \frac{(bc)^n}{1-bc} + \frac{(ca)^n}{1-ca}$$

where $a, b, c \geq 0$ and $a + b + c = 1$.

Answer: When $n = 1$, then maximum is $\frac{3}{8}$. When $n \geq 2$, the answer is $\frac{1}{3 \cdot 4^{n-1}}$.

Solution: First consider the case when $n \geq 2$. By AM-GM, we have $ab \leq \left(\frac{a+b}{2}\right)^2 \leq \frac{1}{4}$, and similarly $ab, bc, ca \leq \frac{1}{4}$. So

$$\frac{(ab)^n}{1-ab} + \frac{(bc)^n}{1-bc} + \frac{(ca)^n}{1-ca} \leq \frac{4}{3}\left((ab)^n + (bc)^n + (ca)^n\right).$$

Thus, we have to prove that $(ab)^n + (bc)^n + (ca)^n \leq \frac{1}{4^n}$. Without loss of generality, suppose that a is the maximum among a, b, c. We have $a(1-a) \leq \frac{1}{4}$, so

$$\frac{1}{4^n} \geq a^n(1-a)^n = a^n(b+c)^n \geq a^n b^n + a^n c^n + n a^n b^{n-1} c \geq a^n b^n + b^n c^n + c^n a^n.$$

So we have proved that in this case the maximum is at most $\frac{1}{3 \cdot 4^{n-1}}$. Furthermore, this bound can be attained when $a = b = \frac{1}{2}$, $c = 0$. Therefore, the maximum value of the expression is $\frac{1}{3 \cdot 4^{n-1}}$ when $n \geq 2$.

Now, suppose that $n = 1$. The value of $\frac{3}{8}$ can be attained by $a = b = c = \frac{1}{3}$. So it remains to prove that

$$\frac{ab}{1-ab} + \frac{bc}{1-bc} + \frac{ca}{1-ca} \geq \frac{3}{8}.$$

Since $a + b + c = 1$, it is equivalent to prove that

$$\frac{ab}{(a+b+c)^2 - ab} + \frac{bc}{(a+b+c)^2 - bc} + \frac{ca}{(a+b+c)^2 - ca} \geq \frac{3}{8}.$$

Now, clearing the denominators and simplifying (details omitted here ...), we see that the above inequality is equivalent to

$$3[6,0,0]+14[5,1,0]+2[4,2,0]+10[4,1,1] \geq 6[3,3,0]+10[3,2,1]+13[2,2,2], \qquad (\dagger)$$

where we adopt the notation

$$[p,q,r]=\sum_{\text{sym}} a^p b^q c^r = a^p b^q c^r + a^p b^r c^q + a^q b^p c^r + a^q b^r c^p + a^r b^p c^q + a^r b^q c^p.$$

Now, (†) is true as it is the sum of the following inequalities, each of which is true due to Muirhead's Inequality:

$$3[6,0,0] \geq 3[3,3,0], \qquad 3[5,1,0] \geq 3[3,3,0], \qquad 10[4,1,1] \geq 10[3,2,1],$$
$$11[5,1,0] \geq 11[2,2,2], \qquad 2[4,2,0] \geq 2[2,2,2].$$

IMO Training Camp Mock Olympiad #4

July 6, 2008

Time limit: 4.5 hours

1. Let $b, n > 1$ be integers. Suppose that for each integer $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

   Solution: See IMO Shortlist 2007 Problem N2

2. Consider those functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy the condition

   $$f(m+n) \geq f(m) + f(f(n)) - 1$$

   for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

   ($\mathbb{N}$ denotes the set of all positive integers.)

   Solution: See IMO Shortlist 2007 Problem A2

3. Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I and F are collinear.

   Solution: See IMO Shortlist 2007 Problem G8

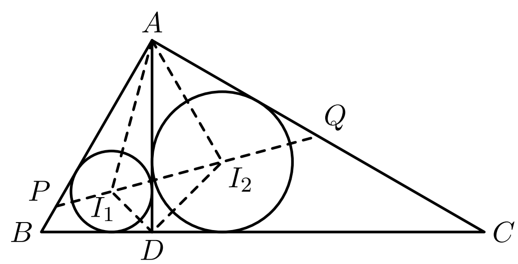
IMO Training Camp Mock Olympiad #5

July 8, 2008

Time limit: 4.5 hours

1. Given non-obtuse triangle ABC, let D be the foot of the altitude from A to BC, and let I_1, I_2 be the incenters of triangles ABD and ACD, respectively. The line I_1I_2 intersects AB and AC at P and Q, respectively. Show that $AP = AQ$ if and only if $AB = AC$ or $\angle A = 90°$.

   Solution: First, we prove that if $AB = AC$ or $\angle A = 90$, then $AP = AQ$. If $AB = AC$, then the result is obvious, so assume that $\angle A = 90°$. Let P', Q' be the points on AB, AC, respectively, such that $AP' = AQ' = AD$, so $\angle AP'Q' = \angle AQ'P' = 45°$. Let I_1' and I_2' be the intersections of the bisectors of $\angle BAD$ and $\angle CAD$, respectively, with $P'Q'$. Then triangles $AP'I_1$ and ADI_1' are congruent, and triangles and $AQ'I_2'$ and ADI_2' are congruent, so $\angle ADI_1' = \angle ADI_2' = 45°$. Therefore I_1' lies on the angle bisector of $\angle BDA$ and I_2' lies on the angle bisector of $\angle CDA$. Hence, I_1' and I_2' are the incenters of ABD and ACD, respectively, so $I_1' = I_1$ and $I_2' = I_2$. This means that $P' = P$ and $Q' = Q$, so $AP = AQ$.



   Conversely, suppose that $AP = AQ$. Let D' be the point on AD such that $AD' = AP = AQ$. If $D \neq D'$, then triangles API_1 and $AD'I_1$ are congruent, and triangles AQI_2 and $AD'I_2$ are congruent, so $\angle I_1D'A = \angle I_2D'A$. Then $\angle I_1D'D = \angle I_2D'D$, and since we also know that $\angle I_1DD' = \angle I_2DD'$ (they are either both $45°$ or both $135°$, depending on whether D' is on the segment AD or the ray AD past D), we have that triangles $I_1D'D$ and $I_2D'D$ are congruent. Thus triangles $AD'I_1$ and $AD'I_2$ are congruent, so in particular $\angle BAD = 2\angle I_1AD' = 2\angle I_2AD' = \angle CAD$, hence triangle ABC is isosceles with $AB = AC$.

   If $D = D'$, then $\angle API_1 = \angle ADI_1 = \angle ADI_2 = \angle AQI_2 = 45°$, so $\angle A = 90°$.

   Comment: Various trigonometric solutions are available, and they are generally pretty easy.

2. Let λ be the positive root of the equation $t^2 - 2008t - 1 = 0$. Define the sequence $x_0, x_1, \ldots$ by setting

   $$x_0 = 1 \quad \text{and} \quad x_{n+1} = \lfloor \lambda x_n \rfloor \text{ for } n \geq 0.$$

   Find the remainder when x_{2008} is divided by 2008.

   Solution: Because $\lambda^2 - 2008 - 1 = 0$, we have $\lambda = 2008 + \frac{1}{\lambda}$. Note also that x_n is an integer. We conclude that

   $$x_{n+1} = \lfloor x_n \lambda \rfloor = \left\lfloor 2008x_n + \frac{x_n}{\lambda} \right\rfloor = 2008x_n + \left\lfloor \frac{x_n}{\lambda} \right\rfloor \equiv \left\lfloor \frac{x_n}{\lambda} \right\rfloor \pmod{2008}.$$

Because $x_n = \lfloor x_{n-1}\lambda \rfloor$ and x_{n-1} is an integer and λ is irrational, we have $x_n = x_{n-1}\lambda - \epsilon$, where $0 < \epsilon < 1$ is the fractional part of $x_{n-1}\lambda$. Since $\lambda > 1$, we have $0 \le \frac{\epsilon}{\lambda} < 1$, and so

$$\left\lfloor \frac{x_n}{\lambda} \right\rfloor = \left\lfloor x_{n-1} - \frac{\epsilon}{\lambda} \right\rfloor = x_{n-1} - 1.$$

It follows that $x_{n+1} \equiv x_{n-1} - 1 \pmod{2008}$. Therefore, by induction $x_{2008} \equiv x_0 - 1004 \equiv 1005 \pmod{2008}$.

3. Let $A_0 = (a_1, \ldots, a_n)$ be a finite sequence of real numbers. For each $k \ge 0$, from the sequence $A_k = (x_1, \ldots, x_n)$ we construct a new sequence A_{k+1} in the following way.

   We choose a partition $\{1, \ldots, n\} = I \cup J$, where I and J are two disjoint sets, such that the expression

   $$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$$

   attains the smallest possible value. (We allow the sets I or J to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily. Then we set $A_{k+1} = (y_1, \ldots, y_n)$, where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

   Prove that for some k, the sequence A_k contains an element x such that $|x| \ge \frac{n}{2}$.

   Solution: See IMO Shortlist 2007 Problem C4

IMO Training Camp Mock Olympiad #6

July 11, 2008

Time limit: 4.5 hours

1. Let ABC be a triangle, and let M be the midpoint of side BC. Triangles ABM and ACM are inscribed in circles ω_1 and ω_2, respectively. Points P and Q are midpoints of arcs AB and AC (not containing M, on ω_1 and ω_2 respectively). Prove that $PQ \perp AM$.

   First solution: By arc midpoints, we have that $PA = PB$ and $QA = QC$, $\angle PMA = \angle PMB$, and $\angle QMA = \angle QMC$. Pick point D such that A, D lie on the same side of M and $MB = MC = MD$. By SAS, $\triangle PAM \cong \triangle PDM$, and $\triangle QMC \cong \triangle QMD$; consequently, $PD = PA$ and $QD = QA$. Therefore, $PDQA$ is a kite, and so $PQ \perp AD \implies PQ \perp AM$.

   Second solution: Consider a rotation about P that brings B to A, and let the image of M be T. Since $\angle PBM + \angle PAM = 180°$, we see that M, A, T are collinear and $MB = AT = MC$.

   Similarly, consider a rotation about Q that brings C to A, this must also bring M to T as the triangles QCM and QAT have equal side lengths.

   Now, notice that in quadrilateral $MPTQ$ we have $PM = PT$ and $QM = QT$, so it is a kite, and thus $PQ \perp MT$. The result follows from the fact that A lies on line MT.

2. Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be two sequences of distinct real numbers such that $a_i + b_j \neq 0$ for all i, j. Show that if

$$\sum_{j=1}^{n} \frac{c_{jk}}{a_i + b_j} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise,} \end{cases}$$

   then

$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} = (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n).$$

   Solution: Let $r_i = \sum_{k=1}^{n} c_{jk}$. Then

$$\sum_{j=1}^{n} \frac{r_j}{a_i + b_j} = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{c_{jk}}{a_i + b_j} = \sum_{k=1}^{n} \sum_{j=1}^{n} \frac{c_{jk}}{a_i + b_j} = 1, \qquad (1)$$

   for all $i = 1, 2, \dots, n$. We wish to determine $r_1 + \cdots + r_n$. Let

$$R(x) = \sum_{j=1}^{n} \frac{r_j}{x + b_j}. \qquad (2)$$

   Then $R(x) = P(x)/Q(x)$ where $Q(x) = (x + b_1)(x + b_2) \cdots (x + b_n)$ and $P(x)$ has degree at most $n - 1$. By (1), $R(a_1) = R(a_2) = \cdots = R(a_n) = 1$, so if we write

$$R(x) = 1 - \frac{S(x)}{Q(x)},$$

then $S(x)$ is a monic polynomials of degree n and $S(a_1) = S(a_2) = \cdots = S(a_n) = 0$. Hence

$$S(x) = (x - a_1)(x - a_2) \cdots (x - a_n).$$

Consider the coefficient of x^{n-1} in $P(x) = Q(x) - S(x)$. Form (2), this coefficient is $r_1 + \cdots + r_n$. On the other hand,

$$Q(x) = (x + b_1)(x + b_2) \cdots (x + b_n) \quad \text{and} \quad S(x) = (x - a_1)(x - a_2) \cdots (x - a_n).$$

From this we see that coefficient x^{n-1} in $Q(x) - S(x)$ equals to $(a_1 + a_2 + \cdots + a_n) + (b_1 + b_2 + \cdots + b_n)$. Hence we have our desired result.

3. Let X be a subset of $\mathbb{Z}$. Denote

$$X + a = \{x + a | x \in X\}.$$

Show that if there exist integers $a_1, a_2, \ldots, a_n$ such that $X + a_1, X + a_2, \ldots, X + a_n$ form a partition of $\mathbb{Z}$, then there is an non-zero integer N such that $X = X + N$.

Solution:

Assume wlog that $0 = a_1 < a_2 < \cdots < a_n$. Define a map $f : \mathbb{Z} \to \{a_1, a_2, \ldots, a_n\}$ thus: for $r \in \mathbb{Z}$, $f(r)$ is the unique a_i such that $r = X + a_i$.

Suppose we are given $f(r), f(r+1), f(r+2), \ldots, f(r+a_n)$ for some $r \in \mathbb{Z}$. Then the value of $f(r + a_n + 1)$ is determined: if there exists $a_i > 0$ such that $f(r + a_n + 1 - a_i) = 0$, then $f(r + a_n + 1) = a_i$; otherwise, $f(r + a_n + 1) = 0 = a_1$. The value of $f(r-1)$ is also determined: if there exists $a_i < a_n$ such that $f(r - 1 + a_n - a_i) = a_n$, then $f(r-1) = a_i$; otherwise, $f(r-1) = a_n$.

By the Pigeonhole principle, there must exist distinct integers r and s such that

$$(f(r), f(r+1), \ldots, f(r + a_n)) = (f(s), f(s+1), \ldots, f(s + a_n)).$$

The preceding paragraph, along with a straightforward induction argument shows that that f is periodic with period $|r - s|$, and therefore $X = X + |r - s|$.

IMO Training Camp Mock Olympiad #7

July 13, 2008

Time limit: 4.5 hours

1. Let T be a finite set of real numbers satisfying the property: For any two elements t_1 and t_2 in T, there is a element t in T such that t_1, t_2, t (not necessarily in that order) are three consecutive terms of an arithmetic sequence. Determine the maximum number of elements T can have.

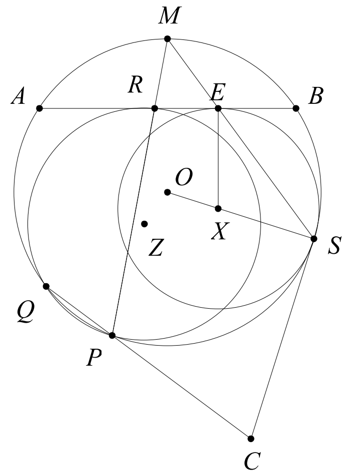
   Solution: The answer is five.

   Let m denote this maximum number. It is not difficult to check that the set $S = \{-3, -1, 0, 1, 3\}$ satisfies the conditions of the problem. Hence $m \geq 5$. It suffices to show that $m \leq 5$. We approach indirectly by assuming on the contrary that $m \geq 6$ and that T is a set satisfies the given property and that T has m elements. Note that adding or multiplying a common number to each element in T will not effect the property of the set. Hence, we may assume that $-3 = t_1 < t_2 < \cdots < t_m = 3$. Then taking $t_1 = -3$ and $t_m = 3$, we know that 0 is an element of T.

   Because T has $m \geq 6$ elements. There is an element in t in T such that t is not an element in S. By symmetry, we may assume that $0 < t < 3$. Because t is not in S, $t \neq 1$. Let t be the element in T such that $t > 0$, $t \neq 1$, and $|t-1|$ is minimal. (This is possible because such t exists by our assumption and that T is finite.) Taking elements -3 and t shows that $\frac{t-3}{2} \in T$. Because $\frac{t-3}{2} < 0$, taking $\frac{t-3}{2}$ and 3 shows that $t' = \frac{1}{2}\left(\frac{t-3}{2} + 3\right) = \frac{t+3}{4} \in T$. Note that $t' > 0$, and that $|t'-1| = \frac{1}{4}|t-1|$, violating the minimality of $|t-1|$. Thus, our assumption was wrong and thus $m = 5$.

2. Let ABM be an isosceles triangle with $AM = BM$. Let O and ω denote the circumcenter and circle of triangle ABM, respectively. Point S and T lie on ω, and tangent lines to ω at S and T meet at C. Chord AB meet segments MS and MT at E and F, respectively. Point X lies on segment OS such that $EX \perp AB$. Point Y lies on segment OT such that $FY \perp AB$. Line ℓ passes through C and intersects ω at P and Q. Chords MP and AB meet R. Let Z denote the circumcenter of triangle PQR. Prove that X, Y, Z are collinear.

   Solution: Consider a homothety centred at S that sends segment OM to segment XE. Let ω_0 be the image of ω under the homothety. We see that ω_0 is centred at X and passes through E and S. Furthermore, because ω is tangent to CS, ω_0 is tangent to CS as well.



Since $\angle MAR = \angle MBA = \angle MPA$, we see that triangles MAR and MPA are similar, from which we get that $MA^2 = MR \cdot MP$. Similarly, triangles MEA and MAS are similar, so $MA^2 = ME \cdot MS$. Thus

$$MR \cdot MP = ME \cdot MS.$$

Also, by Power of a Point, we have

$$CP \cdot CQ = CS^2.$$

It follows that both M and C have equal powers with respect to ω_0 and the circumcircle of PQR. That is, MC is the radical axis of these two circles. Since the radical axis is perpendicular to the line joining the centres of the two circles, we have that $ZX \perp MC$. Similarly, $ZY \perp MC$. Therefore, X, Y, Z are collinear.

Source: China TST 2007

3. Let $a_1, a_2, \ldots$ be a sequence of positive integers satisfying the condition $0 < a_{n+1} - a_n \leq 2008$ for all integers $n \geq 1$ Prove that there exist an infinite number of ordered pairs (p, q) of distinct positive integers such that a_p is a divisor of a_q.

   Solution: Consider all pairs (p, q) of distinct positive integers such that a_p is a divisor of a_q. Assume, by way of contradiction, that there exists a positive N such that $q < N$ for all such pairs.

   We prove by induction on k that for each $k \geq 1$, there exist

   - a finite set $S_k \subset \{a_N, a_{N+1}, \ldots\}$, and
   - a set T_k of 2008 consecutive positive integers greater than or equal to a_N,

   such that at least k elements of T_k are divisible by some element of S_k.

   For $k = 1$, the sets $S_1 = \{a_N\}$ and $T_1 = \{a_N, a_{N+1}, \ldots, a_{N+2007}\}$ suffice.

   Given S_k and T_k (with $k \geq 1$), define

$$T_{k+1} = \{t + \prod_{s \in S_k} s \mid t \in T_k\}.$$

T_{k+1}, like T_k, consists of 2008 consecutive positive integers greater than or equal to a_N — in fact, greater than or equal to $\max S_k$. Also, at least k elements of T_{k+1} are divisible by some element of S_k: namely, $t + \prod_{s \in S_k} s$ for each of the elements $t \in T_k$ which are divisible by some element of S_k.

By the given condition $0 < a_{n+1} - a_n \leq 2008$, and because the elements of T_{k+1} are greater than or equal to a_N, we have that $a_q \in T_{k+1}$ for some $q \geq N$. Because the elements of T_{k+1} are greater than $\max S_k$, we have $a_q \notin S_k$. Thus, by the definition of N, no element of S_k divides a_q.

Hence, at least $k+1$ elements of T_{k+1} are divisible by some element of $S_k \cup \{a_q\}$: at least k elements of T_{k+1} are divisible by some element of S_k, and in addition a_q is divisible by itself. Therefore, setting $S_{k+1} = S_k \cup \{a_q\}$ completes the inductive step.

Setting $k = 2009$, we have the absurd result that T_{2009} is a set of 2008 elements, at least 2009 of which are divisible by some element of S_{2009}. Therefore, our original assumption was false, and for each N there exists $q > N$ and $p \neq q$ such that $a_p \mid a_q$. It follows that there are infinitely many ordered pairs (p, q) with $p \neq q$ and $a_p \mid a_q$.

Source: Vietnam TST 2001