IMO Training Camp Mock Olympiad #2

July 3, 2008

Time limit: 4.5 hours

1. Given an isosceles triangle ABC with $AB = AC$. The midpoint of side BC is denoted by M. Let X be a variable point on the shorter arc MA of the circumcircle of triangle ABM. Let T be the point in the angle domain BMA, for which $\angle TMX = 90^\circ$ and $TX = BX$. Prove that $\angle MTB - \angle CTM$ does not depend on X.

2. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + f(y)) = f(x + y) + f(y)$$

for all $x, y \in \mathbb{R}^+$. (Symbol $\mathbb{R}^+$ denotes the set of all positive real numbers.)

3. For a prime p and a positive integer n, denote by $\nu_p(n)$ the exponent of p in the prime factorization of n. Given a positive integer d and a finite set $\{p_1, \ldots, p_k\}$ of primes. Show that there are infinitely many positive integers n such that $d \mid \nu_{p_i}(n!)$ for all $1 \le i \le k$.

IMO Training Camp Mock Olympiad #3

July 5, 2008

Time limit: 4.5 hours

1. A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that in this dissection, there exists a rectangle having no point on the boundary of the square.

2. The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

3. Let n be a fixed positive integer. Find the maximum value of the expression

$$\frac{(ab)^n}{1-ab} + \frac{(bc)^n}{1-bc} + \frac{(ca)^n}{1-ca}$$

where $a, b, c \geq 0$ and $a + b + c = 1$.

IMO Training Camp Mock Olympiad #4

July 6, 2008

Time limit: 4.5 hours

1. Let $b, n > 1$ be integers. Suppose that for each integer $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.

2. Consider those functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy the condition

$$f(m+n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

($\mathbb{N}$ denotes the set of all positive integers.)

3. Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I and F are collinear.

IMO Training Camp Mock Olympiad #5

July 8, 2008

Time limit: 4.5 hours

1. Given non-obtuse triangle ABC, let D be the foot of the altitude from A to BC, and let I_1, I_2 be the incenters of triangles ABD and ACD, respectively. The line I_1I_2 intersects AB and AC at P and Q, respectively. Show that $AP = AQ$ if and only if $AB = AC$ or $\angle A = 90°$.

2. Let λ be the positive root of the equation $t^2 - 2008t - 1 = 0$. Define the sequence $x_0, x_1, \ldots$ by setting
$$x_0 = 1 \quad \text{and} \quad x_{n+1} = \lfloor \lambda x_n \rfloor \text{ for } n \geq 0.$$
Find the remainder when x_{2008} is divided by 2008.

3. Let $A_0 = (a_1, \ldots, a_n)$ be a finite sequence of real numbers. For each $k \geq 0$, from the sequence $A_k = (x_1, \ldots, x_n)$ we construct a new sequence A_{k+1} in the following way.

   We choose a partition $\{1, \ldots, n\} = I \cup J$, where I and J are two disjoint sets, such that the expression
$$\left| \sum_{i \in I} x_i - \sum_{j \in J} x_j \right|$$
   attains the smallest possible value. (We allow the sets I or J to be empty; in this case the corresponding sum is 0.) If there are several such partitions, one is chosen arbitrarily. Then we set $A_{k+1} = (y_1, \ldots, y_n)$, where $y_i = x_i + 1$ if $i \in I$, and $y_i = x_i - 1$ if $i \in J$.

   Prove that for some k, the sequence A_k contains an element x such that $|x| \geq \frac{n}{2}$.

IMO Training Camp Mock Olympiad #6

July 11, 2008

Time limit: 4.5 hours

1. Let ABC be a triangle, and let M be the midpoint of side BC. Triangles ABM and ACM are inscribed in circles ω_1 and ω_2, respectively. Points P and Q are midpoints of arcs AB and AC (not containing M, on ω_1 and ω_2 respectively). Prove that $PQ \perp AM$.

2. Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be two sequences of distinct real numbers such that $a_i + b_j \neq 0$ for all i, j. Show that if
$$\sum_{j=1}^{n} \frac{c_{jk}}{a_i + b_j} = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{otherwise,} \end{cases}$$
then
$$\sum_{j=1}^{n} \sum_{k=1}^{n} c_{jk} = (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n).$$

3. Let X be a subset of $\mathbb{Z}$. Denote
$$X + a = \{x + a | x \in X\}.$$
Show that if there exist integers $a_1, a_2, \dots, a_n$ such that $X + a_1, X + a_2, \dots, X + a_n$ form a partition of $\mathbb{Z}$, then there is an non-zero integer N such that $X = X + N$.

IMO Training Camp Mock Olympiad #7

July 13, 2008

Time limit: 4.5 hours

1. Let T be a finite set of real numbers satisfying the property: For any two elements t_1 and t_2 in T, there is a element t in T such that t_1, t_2, t (not necessarily in that order) are three consecutive terms of an arithmetic sequence. Determine the maximum number of elements T can have.

2. Let ABM be an isosceles triangle with $AM = BM$. Let O and ω denote the circumcenter and circle of triangle ABM, respectively. Point S and T lie on ω, and tangent lines to ω at S and T meet at C. Chord AB meet segments MS and MT at E and F, respectively. Point X lies on segment OS such that $EX \perp AB$. Point Y lies on segment OT such that $FY \perp AB$. Line ℓ passes through C and intersects ω at P and Q. Chords MP and AB meet R. Let Z denote the circumcenter of triangle PQR. Prove that X, Y, Z are collinear.

3. Let $a_1, a_2, \ldots$ be a sequence of positive integers satisfying the condition $0 < a_{n+1} - a_n \leq 2008$ for all integers $n \geq 1$ Prove that there exist an infinite number of ordered pairs (p, q) of distinct positive integers such that a_p is a divisor of a_q.