# Functional Equations
# IMO Training Camp 2008

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Unless specified otherwise, all functions are real-valued and are defined for all real numbers.

1. Find all solutions of $f(x+y) + f(x-y) = 2f(x)\cos y$.
2. $f(x)$ is defined for $x \neq 0, 1$. Solve the functional equation

$$f(x) + f\left(\frac{1}{1-x}\right) = x$$

3. Find all continuous functions that satisfy

$$f(x+y) = f(x) + f(y) + xy(x+y)$$

4. **IMO 1977** $f : \mathbb{N} \to \mathbb{N}$ is a function satisfying $f(n+1) > f(f(n))$ for all $n$. Prove that $f(n) = n$ for all $n$.
5. Find all $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(m^2 + n) = f(m + n^2)$.
6. Find all continuous functions satisfying $f(x+y) = f(x) + f(y) + f(x)f(y)$.
7. Find all $f : \mathbb{Z} \to \mathbb{Z}$ satisfying $f(x+y) + f(x-y) = 2f(x) + 2f(y)$ for all $x, y \in \mathbb{Z}$.
8. Prove that $f$ is periodic if for fixed $a$ and any $x$:

$$f(x+1) = \frac{1 + f(x)}{1 - f(x)}$$

9. Find all functions from $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which satisfy $f(x,x) = x$, $f(x,y) = f(y,x)$ and $(x+y)f(x,y) = yf(x, x+y)$ for all $x, y \in \mathbb{N}$.
10. The function $f : \mathbb{N} \to \mathbb{N}$ satisfies $f(f(m) + f(n)) = m + n$ for all $m, n \in \mathbb{N}$. Find all possible values of $f(2008)$.
11. Find all functions satisfying for all $x, y \in \mathbb{R}$:

$$xf(y) + yf(x) = (x+y)f(x)f(y)$$

12. Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) + f(f(n)) = 6n$ for all $n \in \mathbb{N}$. Find $f(n)$.
13. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that $f(x+1) = f(x) + 1$ and $f(x^2) = f(x)^2$.
14. Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) + f(f(n)) = 6n$ for all $n \in \mathbb{N}$. Find $f(n)$.

15. Find all functions satisfying $xf(x) + f(1-x) = x^3 - x$ for all $x \in \mathbb{R}$.

16. $f : \mathbb{R}^+ \to \mathbb{R}$ satisfies $f(1) = 1$ and $f(x^2 + y^2) = f(x + y)$ for all $x, y \geq 0$. Prove that $f(x) = 1$ for all $x \geq 0$.

17. Find all functions $f : \mathbb{N} \to \mathbb{Z}^+$ satisfying

$$f(f(f(n))) + f(f(n)) + f(n) = 3n$$

18. The function $f(x)$ is defined for all $x > 0$ and is strictly increasing. Additionally $f(x) > -1/x$ and $f(x)f(f(x) + 1/x) = 1$. Find $f(1)$ and give an example of such a function.

19. Find all continuous functions satisfying $f(x+y)f(x-y) = [f(x)f(y)]^2$.

20. **Balkan 2000** Find all functions satisfying $f(xf(x) + f(y)) = f(x)^2 + y$ for all $x, y \in \mathbb{R}$.

21. **IMO 1968** For some positive constant $a$ let $f$ satisfy the functional equation

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2}$$

Prove that $f$ is periodic and give an example of a non-constant solution for $a = 1$.

22. **IMO 1983** Find all functions $f$ defined for positive real numbers which take positive real values and satisfy the conditions $f(xf(y)) = yf(x)$ for positive $x, y$ and $f(x) \to 0$ as $x \to \infty$.

23. **IMO 1986** Find all functions $f$ defined on non-negative real numbers such that $f(x) \geq 0$ for all $x$, $f(xf(y))f(y) = f(x+y)$, $f(2) = 0$, and $f(x) \neq 0$ for $0 \leq x < 2$.

24. **IMO 1990** Find a function $f : \mathbb{Q}^+ \mapsto \mathbb{Q}^+$ which satisfies $f(xf(y)) = f(x)/y$.

25. **IMO 1992** Find all functions satisfying

$$f(x^2 + f(y)) = y + f(x)^2$$

26. **IMO 1993** Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(1) = 2$, $f(f(n)) = f(n) + n$ and $f(n) < f(n+1)$ for all $n \in \mathbb{N}$.

27. **IMO Shortlist 1995** Does there exist a function $f$ such that $f(x)$ is bounded, $f(1) = 1$ and $f(x + 1/x^2) = f(x) + f(1/x)^2$ for all non-zero $x$?

28. **IMO 1996** Find all functions $f : \{0, 1, \cdots\} \to \{0, 1, \cdots\}$ such that $f(m + f(n)) = f(f(m)) + f(n)$ for all $m, n \geq 0$.

29. **IMO 1999** Find all functions such that $f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$ for all $x, y \in \mathbb{R}$.

30. **IMO 2002** Find all functions such that $(f(x) + f(y))(f(u) + f(v)) = f(xu - yv) + f(xv + yu)$ for all $x, y, u, v$.