Generating Function

IMO training 2008

1. Show
$$\sum_{i=1}^{n}(-1)^i\binom{a}{n-i}\binom{a+i-1}{i}=0$$
2. Show
$$\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}$$
3. Show [1]
$$2^n\binom{n}{n}+2^{n-1}\binom{n+1}{n}+\cdots+\binom{2n}{n}=2^{2n}$$
4. Show
$$\sum_{i=0}^{n}\binom{2i}{i}\binom{2n-2i}{n-i}=4^n$$
5. Show
$$\sum_{k}\binom{n+k}{2k}2^{n-k}=\frac{1+2^{2n+1}}{3}$$
6. Given $f_0=0$, $f_1=1$, $f_{n+2}=f_{n+1}+f_n$ for $n\geq 0$, show that $\sum_{n=0}^{\infty}\frac{f_n}{2^n}=2$.
7. Let $a+b+c=2008$ where a,b,c are non-negative integers. Let S be the sum of abc for each possible a,b,c. Prove that $1004|S$.
8. Let $S=\{1,2,3,\cdots,2008\}$. Let d_1 be the number of subsets of S such that the sum of the elements are 7 mod 32 and d_2 be the number of subsets of S such that the sum of the elements are 14 mod 16. What is $\frac{d_1}{d_2}$?
9. Let (a_n) be an increasing sequence of non-negative integers such that for any non-negative n, there is exactly one pair, (i,j), that satisfies $n=a_i+2a_j$. What is a_{2008}?
10. Let k be a positive integer. Suppose that the integers $1,2,3,\cdots,3k+1$ are written down in random order. What is the probability that at no time during this process, the sum of the integers that have been written up to that time is a positive integer divisible by 3? [2]
11. Let $n\geq 1$ be an integer. A path from $(0,0)$ to (n,n) in the xy plane is a chain of consecutive unit moves either to the right (move denoted by E) or upwards (move denoted by N), all the moves being made inside the half-plane $x\geq y$. A step in a path is the occurence of two consecutive moves of the form EN. Show that the number of paths from $(0,0)$ to (n,n) that contain exactly s steps ($n\geq s\geq 1$) is $\frac{1}{s}\binom{n-1}{s-1}\binom{n}{s-1}$. [3]
12. Evaluate $\sum_{k}\binom{n}{3k}$.
13. Evaluate $\sum_{k}\binom{n}{k}\binom{n-k}{\lfloor\frac{m-k}{2}\rfloor}2^k$. [4]

[1] p24, Challenging Mathematical Problems with Elementary Solutions
[2] Putnam 2007
[3] Shortlist 1999
[4] p159, Generatingfunctionology

14. A partition of n is an increasing sequence of integers with sum n. For example, the partitions of 5 are: 1, 1, 1, 1, 1; 1, 1, 1, 2; 1, 1, 3; 1, 4; 5; 1, 2, 2; and 2, 3. If p is a partition, $f(p)$ = the number of 1s in p, and $g(p)$ = the number of distinct integers in the partition. Show that $\sum f(p) = \sum g(p)$, where the sum is taken over all partitions of n. [5]

15. Let $A_1 = \phi$, $B_1 = \{0\}$, $A_{n+1} = \{x+1 : x \in B_n\}$, $B_{n+1} = (A_n \cup B_n) - (A_n \cap B_n)$. Determine all n such that $B_n = \{0\}$.

16. Show that the number of partitions of n, such that no integer is multiple of d, is equal to the number of partitions of n, such that no integers appear d or more times.

17. Compute $\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

18. Define the sequence a_n by $\sum_{d|n} a_d = 2^n$. Prove that $n|a_n$. [6]

19. Let a mountain of coins be an arrangement coins in rows such that the coins in each row form a single block, and that in all rows (except the bottom row) each coin touches exactly two coins from the row beneath it. How many mountains of coins have exactly k coins in the bottom row? [7]

20. Let S_n be the number of sequences $(a_1, a_2, \cdots, a_n)$ where $a_i \in \{0, 1\}$ in which no six consecutive blocks are equal. Prove that $S_n \to \infty$ when $n \to \infty$. [8]

21. Let p be an odd prime. Prove that $\sum_{0 \le n \le p} \binom{p}{n}\binom{p+n}{n} = 2^p + 1 \pmod{p^2}$. [9]

22. How many ordered pairs (A, B) of subsets of $\{1, 2, ..., 10\}$ can we find such that each element of A is larger than $|B|$ and each element of B is larger than $|A|$? [10]

23. A country has 1-cent, 2-cent, and 3-cent coins only. Show that the number of ways of changing n cents is exactly the integer nearest to $(n+3)^2/12$. [11]

24. Prove that the number of filling $2 \times 2 \times 2n$ box with $1 \times 1 \times 2$ blocks is a perfect square. [12]

25. Evaluate
$$\sum \frac{1}{n_1! n_2! n_3! \cdots n_{1994}! (n_2 + 2n_3 + 3n_4 + \cdots + 1993 n_{1994})!}$$
where n_i are non-negative integers and $n_1 + 2n_2 + 3n_3 + \cdots + 1994 n_{1994} = 1994$. [13]

26. A machine generates 'A', 'B', or 'C' randomly with equal probability. The characters are recorded to form a string. The machine stops when 'ABA' shows up as a substring. What is the expected length of the string?

27. Prove that there are n^{n-2} trees with n labeled vertices.

[5] USAMO 1986, TofT 1984 has a very similar problem too
[6] p103, Problem-Solving Strategies, Generatingfunctionology has the formula for a_n on p62
[7] p38, Generatingfunctionology
[8] Shortlist 1993
[9] Putnam 1991
[10] Putnam 1990
[11] p106, Generatingfunctionology
[12] p229, Problem-Solving Strategies
[13] Vietnam TST 1994