Combinatorics

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A Taste of Algebraic Combinatorics

Problem: (St. Petersburg) Students in a school go for ice cream in groups of at least two. After $k > 1$ groups have gone, every two students have gone together exactly once. Prove that the number of students in the school is at most k.

Solution. Let there be n students. Note if some student went for ice cream only once, then everyone else has to have gone with that student, due to the constraint that every pair of student have gone together exactly once. Furthermore, since each group consists of two students, no other groups can be formed. However, $k > 1$, so this situation cannot occur. Therefore, every student went for ice cream at least twice.

Let us construct **incidence vectors** $\mathbf{v}_i \in \mathbb{R}^k, 1 \leq 1 \leq n$, representing the students. That is, the j-th component of $\mathbf{v}_i$ is 1 if student i went with the j-th group, and 0 otherwise.

The condition that every two students have gone together exactly once translates into the dot product $\mathbf{v}_i \cdot \mathbf{v}_j = 1$ for $i \neq j$. The condition that every student went for ice cream at least twice translates into $|\mathbf{v}_i|^2 \geq 2$.

We want to prove that $n \leq k$. Suppose otherwise. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ consists of at least $k+1$ vectors in $\mathbb{R}^k$, so they must be linearly dependent. It follows that there are real numbers $\alpha_1, \dots, \alpha_n$, not all zero, such that

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n = 0.$$

Let us square the above expression (i.e., taking the dot product with itself). We get

$$0 = \sum_{i=1}^n \alpha_i^2 |\mathbf{v}_i|^2 + 2\sum_{i<j} \alpha_i \alpha_j \mathbf{v}_i \cdot \mathbf{v}_j = \sum_{i=1}^n \alpha_i^2 |\mathbf{v}_i|^2 + 2\sum_{i<j} \alpha_i \alpha_j = \sum_{i=1}^n \alpha_i^2 \left(|\mathbf{v}_i|^2 - 1\right) + \left(\sum_{i=1}^n \alpha_i\right)^2.$$

However, the RHS expression is positive, since $|\mathbf{v}_i|^2 \geq 2$ and some α_i is nonzero. Contradiction. $\square$

If you know a little bit of linear algebra, try the following problem.

1. (Crux 3037) There are 2007 senators in a senate. Each senator has enemies within the senate. Prove that there is a non-empty subset K of senators such that for every senator in the senate, the number of enemies of that senator in the set K is an even number.

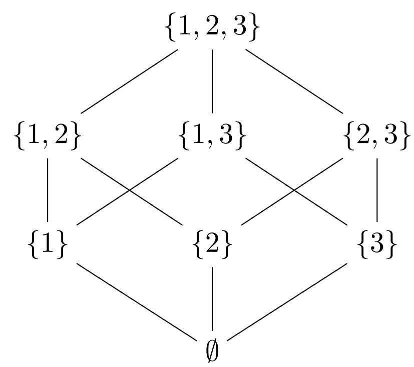
Poset Quickies

Definition. A *partially ordered set* (or *poset* for short) P is a set, also denoted P, together with a binary relation denoted $\leq$ satisfying the following axioms:

- (reflexivity) $x \leq x$ for all $x \in P$
- (antisymmetry) If $x \leq y$ and $y \leq x$, then $x = y$.
- (transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$.

An example of a poset is the set of all subsets of $\{1, 2, \dots, n\}$ under the relation $\subset$. This poset is sometimes called the *Boolean algebra of rank n*, and denoted B_n.

The **Hasse diagram** is a simple way of representing (small) posets. We say that x *covers* y if $x > y$ (i.e., $x \geq y$ and $x \neq y$) and there is no $z \in P$ such that $x > z > y$. If x covers y, then we draw x above y and connect them using a line segment. Note that in general, $x > y$ if and only if x is above y and we trace a downward path from x to y in the Hasse diagram. The Hasse diagram for B_3 is depicted below.



There is one result about posets that has proven useful for olympiad problems. Before we state this result, let us go over some more terminology.

Two elements x, y of a poset are called *comparable* if $x \geq y$ or $x \leq y$, otherwise they are called *incomparable*. A **chain** is a sequence of elements $a_1 < a_2 < \cdots < a_k$, and an **antichain** is a set of pairwise incomparable elements.

Now we are ready to state **Dilworth's Theorem**.

Theorem (Dilworth)**.** Let P be a finite poset. Then the smallest set of chains whose union is P has the same cardinality as the longest antichain.

There is also a dual version of this theorem that's much easier to prove.

Theorem. Let P be a finite poset. Then the smallest set of antichains whose union is P has the same cardinality as the longest chain.

Example: (Romania TST 2005) Let n be a positive integer and S a set of $n^2 + 1$ positive integers with the property that every $(n+1)$-element subset of S contains two numbers one of which is divisible by the other. Show that S contains $n + 1$ different numbers $a_1, a_2, \dots, a_{n+1}$ such that $a_i \mid a_{i+1}$ for each $i = 1, 2, \dots, n$.

Solution. Use the divisibility relation to obtain a poset on S (that is, $x``\leq"y$ iff $x \mid y$. Check that this makes a poset). The condition that there does not exist an $n + 1$ element subset of S that no element divides another translates into the condition that there does not exist an antichain of cardinality $n + 1$ in S. So the longest antichain in S has size at most n, and thus by Dilworth's theorem, S can be written as the union of at most n chains. Since S has $n^2 + 1$ elements, this implies that one of these chains has a cardinality of at least $n + 1$. This implies the result. $\square$

2. (Erdös–Szekeres) Show that any sequence of $ab+1$ real numbers contains either a nondecreasing subsequence of $a+1$ terms, or a nonincreasing subsequence of $b+1$ terms.

3. (Iran 2006) Let k be a positive integer, and let S be a finite collection of intervals on the real line. Suppose that among any $k+1$ of these intervals, there are two with a non-empty intersection. Prove that there exists a set of k points on the real line that intersects with every interval in S.

4. (Slovak competition 2004) Given 1001 rectangles with lengths and widths chosen from the set $\{1, 2, \ldots, 1000\}$, prove that we can chose three of these rectangles, A, B, C, such that A fits into B and B fits into C (rotations allowed).

5. Let G be a simple graph, and let $\chi(G)$ be its chromatic number, i.e., the smallest number of colors needed to color its vertices so that no edge connects two vertices of the same color. Show that there is a path in G of length $\chi(G)$ such that all $\chi(G)$ vertices are of different colors.

Some Challenging Problems

6. Let X be a finite set with $|X| = n$, and let $A_1, A_2, \ldots, A_m$ be three-element subsets of X such that $|A_i \cap A_j| \le 1$ for all $i \neq j$. Show that there exists a subset A of X with at least $\lfloor \sqrt{2n} \rfloor$ elements containing none of the A_i's.

7. (IMO Shortlist 2006) A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

   Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than of arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of *switches*, defined as follows:

   A *switch* consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle.

8. (IMO Shortlist 2006) A *holey triangle* is an upward equilateral triangle of side length n with n upward unit triangular holes cut out. A diamond is a $60^\circ - 120^\circ$ unit rhombus. Prove that a holey triangle T can be tiled with diamonds if and only if the following condition holds: Every upward equilateral triangle of side length k in T contains at most k holes, for $1 \le k \le n$.

9. (IMO Shortlist 2005) There are n markers, each with one side white and the other side black, aligned in a row so that their white sides are up. In each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it and reverse the closest marker to the left and the closest marker to the right of it. Prove that one can achieve the state with only two markers remaining if and only if $n - 1$ is not divisible by 3.

10. (IMO Shortlist 2002) Let $r \ge 2$ be a fixed positive integer, and let $\mathcal{F}$ be an infinite family of sets, each of size r, no two of which are disjoint. Prove that there exists a set of size $r - 1$ that meets each set in $\mathcal{F}$.

11. (Iran 1999) Suppose that $r_1, \ldots, r_n$ are real numbers. Prove that there exists a set $S \subseteq \{1, 2, \ldots, n\}$ such that

$$1 \le |S \cap \{i, i+1, i+2\}| \le 2,$$

   for $1 \le i \le n - 2$, and

$$\left| \sum_{i \in S} r_i \right| \ge \frac{1}{6} \sum_{i=1}^{n} |r_i|.$$

12. Given n collinear points, consider the distances between the points. Suppose each distance appears at most twice. Prove that there are at least $\lfloor n/2 \rfloor$ distances that appear once each.