Selected Geometry Problems from Recent IMO Shortlists

1. (2006/G2) Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $AK/KB = DL/LC$. Suppose that there are points P and Q on the line segment KL satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$. Prove that the points P, Q, B, and C are concyclic.

2. (2006/G3) Let $ABCDE$ be a convex pentagon such that
$$\angle BAC = \angle CAD = \angle DAE \quad \text{and} \quad \angle CBA = \angle DCA = \angle EDA.$$
Diagonals BD and CE meet at P. Prove that line AP bisects side CD.

3. (2004/G2) The circle Γ and the line ℓ do not intersect. Let AB be the diameter of Γ perpendicular to ℓ, with B closer to ℓ than A. An arbitrary point $C \neq A, B$ is chosen on Γ. The line AC intersects ℓ at D. The line DE is tangent to Γ at E, with B and E on the same side of AC. Let BE intersect ℓ at F, and let AF intersect Γ at $G \neq A$. Prove that the reflection of G in AB lies on the line CF.

4. (2006/G4) A point D is chosen on side AC of a triangle ABC with $\angle C < \angle A < 90°$ in such that $BD = BA$. The incircle of triangle ABC is tangent to sides AB and AC at points K and L, respectively. Let J be the incenter of triangle BCD. Prove that line KL intersects segment AJ at its midpoint.

5. (2006/G5) In triangle ABC, let J be the center of excircle tangent to side BC at A_1 and to the extensions of sides AC and AB at B_1 and C_1, respectively. Suppose that lines A_1B_1 and AB are perpendicular and intersect at D. Let E be the foot of the perpendicular from C_1 to line DJ. Prove that $\angle BEA_1 = \angle AEB_1 = 90°$.

6. (2006/G6) Circles ω_1 and ω_2 with respective centers O_1 and O_2 are externally tangent at point D and internally tangent to a circle ω at points E and F, respectively. Line t is the common tangent of ω_1 and ω_2 at D. Let AB be the diameter of ω perpendicular to t, so that A, E and O_1 lie on the same side of t. Prove that lines AO_1, BO_2, EF, and t are concurrent.

7. (2005/G5) Let ABC be an acute-angled triangle with $AB \neq AC$, let H be its orthocentre and M the midpoint of BC. Points D on AB and E on AC are such that $AE = AD$ and D, H, E are collinear. Prove that HM is orthogonal to the common chord of the circumcircles of triangles ABC and ADE.

8. (2004/G5) Let $A_1A_2 \ldots A_n$ be a regular n-gon. The points $B_1, \ldots, B_{n-1}$ are defined as follows:
   - If $i = 1$ or $i = n - 1$, then B_i is the midpoint of the side A_iA_{i+1};
   - If $i \neq 1$, $i \neq n - 1$ and S is the intersection point of A_1A_{i+1} and A_nA_i, then B_i is the intersection point of the bisector of the angle A_iSA_{i+1} with A_iA_{i+1}.

   Prove the equality
$$\angle A_1B_1A_n + \angle A_1B_2A_n + \cdots + \angle A_1B_{n-1}A_n = 180°.$$

9. (2006/G9) Points A_1, B_1 and C_1 are chosen on sides BC, CA, and AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, and CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, and C_2, respectively ($A_2 \neq A, B_2 \neq B$, and $C_2 \neq C$). Points A_3, B_3, and C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of sides BC, CA, and AB, respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

10. (2004/G7) For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.