Inequalities

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1 Classical Theorems

Theorem 1. (**AM-GM**) Let $a_1, \cdots, a_n$ be positive real numbers. Then, we have

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n}.$$

Theorem 2. (**Cauchy-Schwarz**) Let $a_1, \cdots, a_n, b_1, \cdots, b_n$ be real numbers. Then,

$$({a_1}^2 + \cdots + {a_n}^2)({b_1}^2 + \cdots + {b_n}^2) \geq (a_1 b_1 + \cdots + a_n b_n)^2.$$

Theorem 3. (**Jensen**) Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then for any $x_1, x_2, \ldots, x_n \in [a, b]$ and any nonnegative reals $\omega_1, \omega_2, \ldots, \omega_n$ with $\omega_1 + \omega_2 + \cdots + \omega_n = 1$, we have

$$\omega_1 f(x_1) + \omega_2 f(x_2) + \cdots + \omega_n f(x_n) \geq f(\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n).$$

If f is concave, then the inequality is flipped.

Theorem 4. (**Weighted AM-GM**) Let $\omega_1, \cdots, \omega_n > 0$ with $\omega_1 + \cdots + \omega_n = 1$. For all $x_1, \cdots, x_n > 0$, we have

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n \geq x_1^{\omega_1} x_2^{\omega_2} \cdots x_n^{\omega_n}.$$

Theorem 5. (**Schur**) Let x, y, z be nonnegative real numbers. For any $r > 0$, we have

$$\sum_{\text{cyclic}} x^r (x - y)(x - z) \geq 0.$$

Definition 1. (**Majorization**) Let $\mathbf{x} = (x_1, x_2, \ldots x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two sequences of real numbers. Then $\mathbf{x}$ is said to *majorize* $\mathbf{y}$ (denoted $\mathbf{x} \succ \mathbf{y}$) if the following conditions are satisfied

- $x_1 \geq x_2 \geq x_3 \cdots \geq x_n$ and $y_1 \geq y_2 \geq y_3 \cdots \geq y_n$; and
- $x_1 + x_2 + \cdots + x_k \geq y_1 + y_2 + \cdots + y_k$, for $k = 1, 2, \ldots, n - 1$; and
- $x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$.

Theorem 6. (**Muirhead**)[1] Suppose that $(a_1, \ldots, a_n) \succ (b_1, \ldots, b_n)$, and $x_1, \ldots x_n$ are positive real numbers, then

$$\sum_{\text{sym}} x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \geq \sum_{\text{sym}} x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n}.$$

where the symmetric sum is taken over all $n!$ permutations of $x_1, x_2, \ldots, x_n$.

Theorem 7. (**Karamata's Majorization inequality**) Let $f : [a, b] \to \mathbb{R}$ be a convex function. Suppose that $(x_1, \cdots, x_n) \succ (y_1, \cdots, y_n)$, where $x_1, \cdots, x_n, y_1, \cdots, y_n \in [a, b]$. Then, we have

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

[1] Practical note about Muirhead: don't try to apply Muirhead when there are more than 3 variables, since mostly likely you won't succeed (and never, ever try to use Muirhead when the inequality is only cyclic but not symmetric, since it is incorrect to use Muirhead there)

Theorem 8. (**Power Mean**) Let $x_1, \cdots, x_n > 0$. The power mean of order r is defined by

$$M_{(x_1,\cdots,x_n)}(0) = \sqrt[n]{x_1 \cdots x_n}\,, \quad M_{(x_1,\cdots,x_n)}(r) = \left(\frac{x_1^r + \cdots + {x_n}^r}{n}\right)^{\frac{1}{r}} \quad (r \neq 0).$$

Then, $M_{(x_1,\cdots,x_n)} : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and monotone increasing.

Theorem 9. (**Bernoulli**) For all $r \geq 1$ and $x \geq -1$, we have

$$(1+x)^r \geq 1 + rx.$$

Definition 2. (**Symmetric Means**) For given arbitrary real numbers $x_1, \cdots, x_n$, the coefficient of t^{n-i} in the polynomial $(t + x_1) \cdots (t + x_n)$ is called the i-th elementary symmetric function σ_i. This means that

$$(t + x_1) \cdots (t + x_n) = \sigma_0 t^n + \sigma_1 t^{n-1} + \cdots + \sigma_{n-1} t + \sigma_n.$$

For $i \in \{0, 1, \cdots, n\}$, the i-th elementary symmetric mean S_i is defined by

$$S_i = \frac{\sigma_i}{\binom{n}{i}}.$$

Theorem 10. Let $x_1, \ldots, x_n > 0$. For $i \in \{1, \cdots, n\}$, we have

(1) (**Newton**) $\frac{S_i}{S_{i+1}} \geq \frac{S_{i-1}}{S_i}$,
(2) (**Maclaurin**) ${S_i}^{\frac{1}{i}} \geq {S_{i+1}}^{\frac{1}{i+1}}$.

Theorem 11. (**Rearrangement**) Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. For any permutation σ of $\{1, \ldots, n\}$, we have

$$\sum_{i=1}^{n} x_i y_i \geq \sum_{i=1}^{n} x_i y_{\sigma(i)} \geq \sum_{i=1}^{n} x_i y_{n+1-i}.$$

Theorem 12. (**Chebyshev**) Let $x_1 \geq \cdots \geq x_n$ and $y_1 \geq \cdots \geq y_n$ be real numbers. We have

$$\frac{x_1 y_1 + \cdots + x_n y_n}{n} \geq \left(\frac{x_1 + \cdots + x_n}{n}\right)\left(\frac{y_1 + \cdots + y_n}{n}\right).$$

Theorem 13. (**Hölder**)[2] Let $x_1, \cdots, x_n, y_1, \cdots, y_n$ be positive real numbers. Suppose that $p > 1$ and $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then, we have

$$\sum_{i=1}^{n} x_i y_i \leq \left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} {y_i}^q\right)^{\frac{1}{q}}.$$

More generally, let x_{ij} $(i = 1, \cdots, m, j = 1, \cdots n)$ be positive real numbers. Suppose that $\omega_1, \cdots, \omega_n$ are positive real numbers satisfying $\omega_1 + \cdots + \omega_n = 1$. Then, we have

$$\prod_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij}\right)^{\omega_j} \geq \sum_{i=1}^{m} \left(\prod_{j=1}^{n} {x_{ij}}^{\omega_j}\right).$$

Theorem 14. (**Minkowski**)[3] If $x_1, \cdots, x_n, y_1, \cdots, y_n > 0$ and $p > 1$, then

$$\left(\sum_{i=1}^{n} {x_i}^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} {y_i}^p\right)^{\frac{1}{p}} \geq \left(\sum_{i=1}^{n} (x_i + y_i)^p\right)^{\frac{1}{p}}$$

[2] Think of this as generalized Cauchy, as you can use it for more than two sequences.
[3] Think of this as generalized triangle inequality.

Quotient Substitution

1. Let $a, b, c > 0$ with $abc = 1$. Prove that
$$\frac{a}{ab+1} + \frac{b}{bc+1} + \frac{c}{ca+1} \geq \frac{3}{2}.$$

2. (Russia 2004) Let $n > 3$ and $x_1, x_2, \ldots x_n > 0$ with $x_1 x_2 \ldots x_n = 1$. Prove that
$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \cdots + \frac{1}{1+x_n+x_nx_1} > 1.$$

3. Let $a, b, c, d > 0$ with $abcd = 1$. Prove that
$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+d)} + \frac{1}{d(1+a)} \geq 2.$$

4. (Crux 3147) Let $n \geq 3$, $x_1, \ldots, x_n > 0$ such that $x_1 x_2 \cdots x_n = 1$. For $n = 3$ and $n = 4$ prove that
$$\frac{1}{x_1^2 + x_1x_2} + \frac{1}{x_2^2 + x_2x_3} + \cdots + \frac{1}{x_n^2 + x_nx_1} \geq \frac{n}{2}.$$

5. (IMO 2000) Let $a, b, c > 0$ with $abc = 1$. Prove that
$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Cauchy

6. Let $a, b, x, y, z > 0$. Prove that
$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} \geq \frac{3}{a+b}.$$

7. Let $a, b, c, x, y, z \geq 0$. Prove that
$$(b+c)x + (c+a)y + (a+b)z \geq 2\sqrt{(xy+yz+zx)(ab+bc+ca)}$$

8. (IMO 1995) Let $a, b, c > 0$ with $abc = 1$. Prove that
$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Convexity and Endpoints

9. Let $0 \leq a, b, c, d \leq 1$. Prove that
$$(1-a)(1-b)(1-c)(1-d) + a + b + c + d \geq 1.$$

10. If $a, b, c, d, e \in [p, q]$ with $p > 0$, prove that
$$(a+b+c+d+e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \leq 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2.$$

11. Let σ_i denote the i-th elementary symmetric polynomial in $x_1, x_2, \ldots, x_n$. That is,
$$(t+x_1)(t+x_2)\cdots(t+x_n) = t^n + \sigma_1 t^{n-1} + \sigma_2 t^{n-2} + \cdots + \sigma_{n-1} t + \sigma_n.$$
Let $a_0, a_1, \ldots, a_n$ be real numbers. Prove that the maximum and minimum values of the expression
$$a_0 + a_1\sigma_1 + a_2\sigma_2 + \cdots + a_n\sigma_n$$
under the constraints $x_1, x_2, \ldots, x_n \geq 0$ and $x_1 + x_2 + \cdots + x_n = 1$, can be attained when $(x_1, x_2, \ldots, x_n)$ is of the form $\left(\frac{1}{k}, \frac{1}{k}, \ldots, \frac{1}{k}, 0, 0, \ldots, 0\right)$ for some $1 \leq k \leq n$.

12. (IMO 1984) Let $x, y, z \geq 0$ with $x + y + z = 1$. Prove that
$$0 \leq yz + zx + xy - 2xyz \leq \frac{7}{27}.$$

13. (IMO Shortlist 1993) Let $a, b, c, d \geq 0$ with $a + b + c + d = 1$. Prove that
$$abc + abd + acd + bcd \leq \frac{1}{27} + \frac{176}{27}abcd.$$

More Inequalities

14. (IMO Shortlist 1986) Find the minimum value of the constant c such that for any $x_1, x_2, \cdots > 0$ for which $x_{k+1} \geq x_1 + x_2 + \cdots + x_k$ for any k, the inequality
$$\sqrt{x_1} + \sqrt{x_2} + \cdots + \sqrt{x_n} \leq c\sqrt{x_1 + x_2 + \cdots + x_n}$$
also holds for any n.

15. (Russia 2002) Let $a, b, c, x, y, z > 0$ with $a + x = b + y = c + z = 1$. Prove that
$$(abc + xyz)\left(\frac{1}{ay} + \frac{1}{bz} + \frac{1}{cx}\right) \geq 3.$$

16. (Korea 2001) Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$ with $x_1^2 + x_2^2 + \cdots + x_n^2 = y_1^2 + y_2^2 + \cdots + y_n^2 = 1$. Prove that
$$(x_1 y_2 - x_2 y_1)^2 \leq 2\left(1 - \sum_{k=1}^{n} x_k y_k\right).$$

17. (USAMO 2004) Let $a, b, c > 0$. Prove that
$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

18. Let $a, b, c > 0$. Prove that
$$\frac{a + \sqrt{ab} + \sqrt[3]{abc}}{3} \leq \sqrt[3]{a \cdot \left(\frac{a+b}{2}\right) \cdot \left(\frac{a+b+c}{3}\right)}.$$

19. (Singapore TST) Let $a_1, \ldots, a_n, b_1, b_2, \ldots, b_n \in [1001, 2002]$, such that $a_1^2 + a_2^2 + \cdots + a_n^2 = b_1^2 + b_2^2 + \cdots + b_n^2$. Show that
$$\frac{a_1^3}{b_1} + \frac{a_2^3}{b_2} + \cdots + \frac{a_n^3}{b_n} \leq \frac{17}{10}(a_1^2 + a_2^2 + \cdots + a_n^2).$$

20. Let $x_1, x_2, \ldots, x_n > 0$ with $x_1 x_2 \cdots x_n = 1$. Prove that
$$\sum_{i=1}^{n} x_i^2 \leq n + \sum_{1 \leq i < j \leq n} (x_i - x_j)^2.$$

21. Prove that if $a_1, a_2, \ldots, a_n \in \mathbb{R}$, then

$$\max_{x\in[0,2]} \prod_{i=1}^{n} |x - a_i| \leq 12^n \max_{x\in[0,1]} \prod_{i=1}^{n} |x - a_i|.$$

22. (Romanian TST 2004) Let $a_1, a_2, \ldots, a_n \in \mathbb{R}$ and S a nonempty subset of $\{1, 2, \ldots, n\}$. Prove that

$$\left(\sum_{i\in S} a_i\right)^2 \leq \sum_{1\leq i\leq j\leq n} (a_i + \cdots + a_j)^2.$$